Integrability vs Supersymmetry:
Poisson Structures of the Pohlmeyer Reduction

David M. Schmidtt[*]

Instituto de Física Teórica - IFT/UNESP
Rua Dr. Bento Teobaldo Ferraz, 271, Bloco II
CEP 01140-070, São Paulo, SP, Brasil.

November 18, 2011

Abstract

We construct recursively an infinite number of Poisson structures for the supersymmetric integrable hierarchy governing the Pohlmeyer reduction of superstring sigma models on the target spaces $AdS_n \times S^n$, $n = 2, 3, 5$. These Poisson structures are all non-local and not relativistic except one, which is the canonical Poisson structure of the semi-symmetric space sine-Gordon model (SSSSG). We verify that the superposition of the first three Poisson structures corresponds to the canonical Poisson structure of the reduced sigma model. Using the recursion relations we construct commuting charges on the reduced sigma model out of those of the SSSSG model and in the process we explain the integrable origin of the Zukhovsky map and the twisted inner product used in the sigma model side. Then, we compute the complete Poisson superalgebra for the conserved Drinfeld-Sokolov supercharges associated to an exotic kind of extended non-local rigid 2d supersymmetry recently introduced in the SSSSG context. The superalgebra has a kink central charge which turns out to be a generalization to the SSSSG models of the well-known central extensions of the $N = 1$ sine-Gordon and $N = 2$ complex sine-Gordon model Poisson superalgebras computed from 2d superspace. The computation is done in two different ways concluding the proof of the existence of 2d supersymmetry in the reduced sigma model phase space under the boost invariant SSSSG Poisson structure.

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[*]david.schmidtt@gmail.com

1 Introduction.

Since the work of Grigoriev and Tseytlin [1], devoted to the study of the classical Pohlmeyer reduction of the Green-Schwarz superstring sigma model (GSsσ) on $AdS_5 \times S^5$, there has been a relatively intense activity focused on the study of the properties of a family of 2d integrable field theories models that appear in the reduction process [2]-[17]. The equivalent integrable field theories that are left after reduction of string and superstring sigma models are known as Symmetric Space Sine-Gordon (SSSG) and Semi-Symmetric Space Sine-Gordon (SSSSG) models, respectively. There are two main motivations for studying these models at classical level: the first one is because the reduced models possesses a manifestly Lorentz invariant integrable hierarchy structure amenable of quantization and the second one is because of the possibility of having 2d world-sheet rigid supersymmetry. At quantum level however, the motivation is even stronger and is essentially based on the possibility of having an eventual first-principles solution of the GSsσ model based on quantum integrability, at least for the GSsσ model on $AdS_5 \times S^5$, in which the Pohlmeyer reduction is expected to survive.

This paper is a continuation of the study of the on-shell 2d supersymmetry properties of the SSSSG models initiated in [15], based on the fermionic symmetry flows approach gradually developed in [18],[19],[20],[15]. The main results of the present paper are the following:

- There is an infinite number of Poisson bi-vectors $\Theta^{(n)}$, $n \in \mathbb{Z}$ on the reduced phase space $\mathcal{P}$ of the GSsσ model on the semi-symmetric space F/G in terms of which the evolution of φ can be written as follows

$$\frac{\partial \varphi}{\partial t_a} = \{\varphi, H_{z^{-4n}a}\}_{-n} = ... = \{\varphi, H_{z^{-4}a}\}_{-1} = \{\varphi, H_a\}_0 = \{\varphi, H_{z^4 a}\}_1 = ... = \{\varphi, H_{z^{4n}a}\}_n, \tag{1}$$

where z is the spectral parameter and $a \subset \widehat{\mathfrak{f}}^{\perp} \subset \widehat{\mathfrak{f}}$ belongs to an affinization of the Lie superalgebra $\mathfrak{f}$.

- The reduced canonical Poisson structure Θ_σ of the GSsσ model can be written as follows

$$\Theta_\sigma = \Theta^{(-1)} - 2\Theta^{(0)} + \Theta^{(1)}. \tag{2}$$

This result was already obtained in [23] for the GSsσ model on $AdS_5 \times S^5$ but here we use different arguments based on integrability which are also valid for the targets $AdS_n \times S^n$, $n = 2, 3$.

- The recursion relations (1) and the latter expression for Θ_σ, i.e, (2) imply the following relation between the commuting charges of the GSsσ models q_σ and the SSSSG models q

$$q_\sigma(a) = \int_{-\infty}^{+\infty} dx \, \langle a, \mathfrak{d} \rangle_\phi \,, \qquad q(a) = \int_{-\infty}^{+\infty} dx \, \langle a, \mathfrak{d} \rangle \,, \qquad \phi(z) = z^{-4} - 2 + z^4,$$

where $\mathfrak{d}$ is a density, $\langle X, Y \rangle$ and $\langle X, Y \rangle_\phi$ are inner products on $\widehat{\mathfrak{f}}$ and $\langle X, Y \rangle_\phi$ is the twisting of $\langle X, Y \rangle$ by the Zukhovsky map $z \to u(z)$.

- The reduced phase space of the GSsσ model on $AdS_n \times S^n$, $n = 2, 3, 5$ is supersymmetric under the Poisson structure $\Theta^{(0)}$. The mixed Poisson bracket computing the central charge is

$$\{q(\epsilon), \overline{q}(\overline{\epsilon})\} = Z, \qquad Z = Str\left(\gamma \epsilon \gamma^{-1} \overline{\epsilon}\right) \big|_{-\infty}^{+\infty},$$

where γ is the fundamental bosonic field of the SSSSG model. This Z generalizes the central charges of the $N = 1$ sine-Gordon and $N = 2$ complex sine-Gordon models.

With these results we conclude the proof of the existence of on-shell rigid 2d supersymmetry on the reduced models.

The outline of the paper is as follows. In section 2 we review the basic properties and results of the integrable hierarchy of the SSSSG models. In section 3 we construct recursively the Poisson structures focusing mainly in the first three, which are constructed explicitly. In section 4 we show several connections between the GSsσ models and the SSSSG models, e.g, the Poisson structures, how to extract the Lax pair of the SSSSG model from that of the GSsσ model, etc. In section 5 we re-study the supersymmetry flow variations and deduce a set of local transformations closing on a superalgebra with field dependent parameters. In section 6 we construct the moment maps associated to the supersymmetry showing, in a different way in contrast to [15], that they are Hamiltonian flows on the reduced phase space. In section 7 we compute in two different ways the mixed Poisson bracket and the kink central charge of the Poisson supersymmetry algebra of the reduced models. Finally, we make some concluding remarks. For the sake of completeness and readability we have tried to be as self-contained as possible.

2 Essentials of the SSSSG integrable hierarchy.

In this section we recall some of the definitions and results of the integrable hierarchy governing the Pohlmeyer reduction of superstring sigma models that we need in the following. The supersymmetric integrable hierarchy, the non-local supersymmetry variations and their associated fermionic conserved charges associated to all the semi-symmetric superspaces involved in the reduction of the GSsσ model on $AdS_n \times S^n$, $n = 2, 3, 5$ was initially introduced in [15] and subsequently rephrased and nicely applied to the $AdS_5 \times S^5$ case in [16]. However, we will use the notation of [16] for the purpose of notational unification.

Considering a finite dimensional real Lie superalgebra $\mathfrak{f}$ endowed with an order four linear automorphism Ω, $\Omega : \mathfrak{f} \to \mathfrak{f}$, $\Omega([X, Y]) = [\Omega(X), \Omega(Y)]$, $\Omega^4 = I$. The superalgebra $\mathfrak{f}$ then admits a $\mathbb{Z}_4$ grade space decomposition satisfying

$$\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3, \qquad \Omega(\mathfrak{f}_j) = (i)^j \mathfrak{f}_j, \qquad [\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{(i+j) \bmod 4}. \tag{3}$$

The even subalgebra is $\mathfrak{f}_{even} = \mathfrak{f}_0 \oplus \mathfrak{f}_2$ while the odd part of $\mathfrak{f}$ is formed by $\mathfrak{f}_{odd} = \mathfrak{f}_1 \oplus \mathfrak{f}_3$.

We need to introduce a semisimple element $\Lambda \in \mathfrak{f}_2$ inducing the following superalgebra splitting

$$\mathfrak{f} = \ker(ad(\Lambda)) \oplus \mathrm{Im}(ad(\Lambda)) \equiv \mathfrak{f}^{\perp} \oplus \mathfrak{f}^{\|}, \qquad \mathfrak{f}^{\perp} \cap \mathfrak{f}^{\|} = \oslash,$$

and restrict ourselves to the situation in which $\mathfrak{f}$ admits an extra $\mathbb{Z}_2$ gradation $\sigma : \mathfrak{f} \to \mathfrak{f}$, $\sigma\left([X,Y]\right) = \left[\sigma\left(X\right), \sigma\left(Y\right)\right]$, $\sigma^2 = I$ with $\sigma(\mathfrak{f}^{\perp}) = \mathfrak{f}^{\perp}$ and $\sigma(\mathfrak{f}^{\|}) = -\mathfrak{f}^{\|}$, implying that $\mathfrak{f}$ is also a symmetric space

$$\left[\mathfrak{f}^{\perp}, \mathfrak{f}^{\perp}\right] \subset \mathfrak{f}^{\perp}, \qquad \left[\mathfrak{f}^{\perp}, \mathfrak{f}^{\|}\right] \subset \mathfrak{f}^{\|}, \qquad \left[\mathfrak{f}^{\|}, \mathfrak{f}^{\|}\right] \subset \mathfrak{f}^{\perp}. \tag{4}$$

We are mainly interested in the cases $\mathfrak{f} = \mathfrak{psu}(n,n \mid n)$ for $n = 1, 2$ and it follows from string theory arguments [1] that it is possible to choose Λ, in an $n \times n$ dimensional supermatrix representation of $\mathfrak{su}(n,n \mid n)$, as follows

$$\Lambda = \frac{i}{2} diag\left(\lambda, \lambda\right), \qquad \lambda = -diag\left(I_n, -I_n\right).$$

This Λ satisfies

$$\Lambda \mathfrak{f}^{\perp} = \mathfrak{f}^{\perp} \Lambda, \qquad \Lambda \mathfrak{f}^{\|} = -\mathfrak{f}^{\|} \Lambda, \qquad -4\Lambda^2 = I_{4n}$$

and in terms of it the projection operators along $\mathfrak{f}^{\perp}$ and $\mathfrak{f}^{\|}$ are given by

$$\pi^{\perp}(*) = -\left\{\Lambda, \{\Lambda, *\}\right\}, \qquad \pi^{\|}(*) = -\left[\Lambda, [\Lambda, *]\right].$$

The connection with the Green-Schwarz superstring sigma models on $AdS_n \times S^n$, $n = 2, 3, 5$ involve respectively, the following [1],[2],[30] semi-symmetric spaces F/G

$$\frac{F}{G}: \quad \frac{PSU(1,1 \mid 2)}{SO(1,1) \times SO(2)}, \qquad \frac{PSU(1,1 \mid 2) \times PSU(1,1 \mid 2)}{SU(1,1) \times SU(2)}, \qquad \frac{PSU(2,2 \mid 4)}{SO(1,4) \times SO(5)},$$

where $G = \exp \mathfrak{g}$, $\mathfrak{g} \equiv \mathfrak{f}_0$, while the connection with the SSSSG models obtained after Pohlmeyer reduction involve the special coset spaces $G/H \oplus \left(\mathfrak{f}_1^{\|} \oplus \mathfrak{f}_3^{\|}\right)$ with

$$\frac{G}{H}: \quad U(1) \times U(1), \qquad \frac{SU(1,1) \times SU(2)}{U(1) \times U(1)}, \qquad \frac{SO(1,4) \times SO(5)}{SU(2)^{\times 4}},$$

where $H = \exp \mathfrak{h}$, $\mathfrak{h} \equiv \mathfrak{f}_0^{\perp}$. These reduced models exhibits and exotic kind of 2d rigid supersymmetry of the type $N = (2,2)$, $N = (4,4)$ and $N = (8,8)$ with R-symmetry group H and where the number of chiral supersymmetries is determined by $N = \dim \mathfrak{f}_{1,3}^{\perp}$, see for instance [15],[16]. We will soon recall how this supersymmetries appear in our construction after affinization of $\mathfrak{f}^{\perp}$.

The algebraic structure underlying the SSSSG integrable hierarchy, is defined by the following twisted loop Lie superalgebra

$$\widehat{\mathfrak{f}} = \bigoplus_{n \in \mathbb{Z} = -\infty}^{+\infty} \left(z^{4n} \otimes \mathfrak{f}_0 + z^{4n+1} \otimes \mathfrak{f}_1 + z^{4n+2} \otimes \mathfrak{f}_2 + z^{4n+3} \otimes \mathfrak{f}_3\right), \tag{5}$$

which can be rewritten as an integer decomposition

$$\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}^{\perp} \oplus \widehat{\mathfrak{f}}^{\|} = \bigoplus_{r \in \mathbb{Z} = -\infty}^{+\infty} \widehat{\mathfrak{f}}_r, \qquad \left[Q, \widehat{\mathfrak{f}}_r\right] = r\widehat{\mathfrak{f}}_r \tag{6}$$

in terms of the homogeneous gradation operator $Q_H \equiv z\frac{d}{dz}$. The kernel subalgebra $\widehat{\mathfrak{f}}^{\perp}$ decomposes as[1]

$$\widehat{\mathfrak{f}}^{\perp} = \widehat{\mathfrak{c}} \ltimes \widehat{\mathfrak{z}}, \qquad \widehat{\mathfrak{c}} = \left[\widehat{\mathfrak{f}}^{\perp}, \widehat{\mathfrak{f}}^{\perp}\right], \qquad \widehat{\mathfrak{z}} = cent(\widehat{\mathfrak{f}}^{\perp}), \tag{7}$$

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[1] The symbol $\ltimes$ denotes central extension.

where $\widehat{\mathfrak{c}}$ and $\widehat{\mathfrak{z}}$ are the commutant part and the center of $\widehat{\mathfrak{f}}^\perp$, respectively. The inner product in $\widehat{\mathfrak{f}}$ is to be defined by[2]

$$\langle X, Y \rangle \equiv \oint \frac{dz}{2\pi i} \frac{1}{z} Str\left(X(z), Y(z) \right) \tag{8}$$

and selects the term of zero Q_H grade, i.e, z^0. Below, we will show that in order to describe conserved quantities in the sigma model side we have to twist this inner product by means of the Zukhovsky map.

The complex variable z enter in the Lax operators as the spectral parameter and it is important to notice that for the cases of interest we have two possible superalgebra isomorphisms of $\widehat{\mathfrak{f}}$, namely

$$z^{\pm 2n}\, \widehat{\mathfrak{f}} \simeq \widehat{\mathfrak{f}}, \qquad z^{\pm 4n}\, \widehat{\mathfrak{f}} \simeq \widehat{\mathfrak{f}}, \qquad n \in \mathbb{Z}. \tag{9}$$

The first isomorphism holds in the case of superstrings on $AdS_3 \times S^3$ because of the identifications $\mathfrak{f}_0 = \mathfrak{f}_2$ and $\mathfrak{f}_1 = \mathfrak{f}_3$, while the other holds for the cases of superstrings on $AdS_2 \times S^2$ and $AdS_5 \times S^5$ and it is because the twisted nature of $\widehat{\mathfrak{f}}$, in contrast to the KdV hierarchy, that the first Poisson structures of the SSSSG hierarchy becomes non-local. See, section 3 below.

The phase space of the SSSSG integrable hierarchy and the symmetry flows of the dynamical system are defined by intersecting the following two co-adjoint orbits of the dressing groups $(\chi, \gamma^{-1}\widetilde{\chi})$, namely $\Xi_a(\chi) = \Xi_a(\gamma^{-1}\widetilde{\chi})$, $a \in \widehat{\mathfrak{f}}^\perp$, with

$$\Xi_a(\chi) \equiv \mathcal{L}_a = \chi\left(\partial_{t_a} + a\right)\chi^{-1} \in \widehat{\mathfrak{f}}^*, \qquad \Xi_a(\gamma^{-1}\widetilde{\chi}) \equiv \gamma^{-1}\mathcal{L}'_a\gamma = \gamma^{-1}\widetilde{\chi}\left(\partial_{t_a} + a\right)\widetilde{\chi}^{-1}\gamma \in \widehat{\mathfrak{f}}^*, \tag{10}$$

where $\gamma \in \exp\mathfrak{g}$, $a \in \widehat{\mathfrak{f}}^\perp$ is an element of Q_H grade $n \in \mathbb{Z}$, t_a is the time variable associated to a and[3]

$$
\begin{aligned}
\chi &= \Phi\Omega\overline{u}^{-1} \in \exp\widehat{\mathfrak{f}}_{\leq 0}, & \Phi(z) &= e^{y(z)} \in \exp\widehat{\mathfrak{f}}_{<0}^{\parallel}, & \Omega(z) &= e^{\theta(z)} \in \exp\widehat{\mathfrak{f}}_{<0}^{\perp}, & \overline{u} &\in \exp\mathfrak{f}_0^\perp, \\
\widetilde{\chi} &= \widetilde{\Phi}\widetilde{\Omega}u^{-1} \in \exp\widehat{\mathfrak{f}}_{\geq 0}, & \widetilde{\Phi}(z) &= e^{\widetilde{y}(z)} \in \exp\widehat{\mathfrak{f}}_{>0}^{\parallel}, & \widetilde{\Omega}(z) &= e^{\widetilde{\theta}(z)} \in \exp\widehat{\mathfrak{f}}_{>0}^{\perp}, & u &\in \exp\mathfrak{f}_0^\perp,
\end{aligned} \tag{11}
$$

are the dressing matrices. We identify $\widehat{\mathfrak{f}} \sim \widehat{\mathfrak{f}}^*$ under the inner product (8).

The world-sheet light-cone coordinates are associated to the first two isospectral times $t_{z^{\pm 2}\Lambda} = x^\pm$, $z^{\pm 2}\Lambda \in \widehat{\mathfrak{z}}$ leading to the following Lax operators[4]

$$\mathcal{L}_\pm = \chi\left(\partial_\pm - z^{\pm 2}\Lambda\right)\chi^{-1} = \gamma^{-1}\mathcal{L}'_\pm\gamma = \gamma^{-1}\widetilde{\chi}\left(\partial_\pm - z^{\pm 2}\Lambda\right)\widetilde{\chi}^{-1}\gamma. \tag{12}$$

The symmetries of the system are introduced through the field variations induced by the trivial relations $[\mathcal{L}_a, \mathcal{L}_\pm] = 0$, $a \in \widehat{\mathfrak{c}}$ and are described by the non-Abelian times t_a. As a consequence, the symmetry variations, when $u = \overline{u} = I$, form a non-Abelian algebra of flows $\widehat{\mathfrak{S}}$ isomorphic to $\widehat{\mathfrak{f}}^\perp$, namely $a \in \widehat{\mathfrak{f}}^\perp \to \partial_{t_a} \equiv \delta_a \in \widehat{\mathfrak{S}}$ with

$$\left[\delta_a, \delta_{a'}\right](*) = \delta_{[a,a']}(*). \tag{13}$$

The affine algebra $\widehat{\mathfrak{f}}^\perp$ is infinite dimensional but includes a very special finite dimensional sub-superalgebra [15],[16] $\widehat{\mathfrak{s}} \subset \widehat{\mathfrak{f}}^\perp$, which is spanned by

$$\widehat{\mathfrak{s}} = \left(z^{-1}\mathfrak{f}_3^\perp \oplus \mathfrak{h} \oplus z\mathfrak{f}_1^\perp\right) \ltimes \mathbb{R}^2, \qquad \mathbb{R}^2 = z^2\Lambda \oplus z^{-2}\Lambda \tag{14}$$

and under (13) turns out to be isomorphic to the following double central extended superalgebra $\mathfrak{s} \simeq \widehat{\mathfrak{s}}$ with[5]

$$\mathfrak{s} = \mathfrak{a} \ltimes \mathbb{R}^2, \qquad \mathfrak{a} = \mathfrak{h} \oplus \mathfrak{f}_1^\perp \oplus \mathfrak{f}_3^\perp, \qquad \mathbb{R}^2 = \partial_+ \oplus \partial_-. \tag{15}$$

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[2]We assume the existence of a supermatrix representation for $\mathfrak{f}$.

[3]The notation $\widehat{\mathfrak{f}}_{<n}, \widehat{\mathfrak{f}}_{>n}, \widehat{\mathfrak{f}}_{\leq n}, \widehat{\mathfrak{f}}_{\geq n}$ stands for an expansion in powers of the spectral parameter z with grade Q_H with values $< n, > n, \leq n$ and $\geq n$.

[4]In what follows we use $x^\pm = t \pm x$ and $a_\pm = \frac{1}{2}(a_0 \pm a_1)$.

[5]Actually, the supersymmetry algebra of the reduced models is $\mathfrak{so}(1,1) \ltimes \left(\mathfrak{a} \ltimes \mathbb{R}^2\right)$, where $\ltimes$ denotes semi-direct sum. We are dropping the Lorentz group $\mathfrak{so}(1,1)$ in $\mathfrak{s}$ because we are also dropping the grading operator Q_H in $\widehat{\mathfrak{s}}$, which is its equivalent.

In particular, for $AdS_n \times S^n$ with $n = 2, 3, 5$ we have, respectively,

$$\mathfrak{a}: \quad \mathfrak{psu}(1\,|\,1)^{\oplus 2}, \qquad \left(\mathfrak{u}(1) \rtimes \mathfrak{psu}(1\,|\,1)^{\oplus 2}\right)^{\oplus 2} \ltimes \mathfrak{u}(1), \qquad \mathfrak{psu}(2\,|\,2)^{\oplus 2}.$$

Recently [8],[17] the q-deformation $\mathfrak{s} \to \mathfrak{s}_q$, with $q = q(k)$, k the level of the WZNW model, have been identified as the deformed supersymmetries of the quantum S-matrix of the Pohlmeyer reduced models. Needless to say, these superalgebras are of the extended type (N, N) with $N = 2, 4$ and 8 with R-symmetry groups $\varnothing$, $U(1)^{\times 2}$ and $SU(2)^{\times 4}$, respectively.

The pre-potentials u and $\overline{u}$ dress the $\partial_\pm$ derivatives into the covariant derivatives

$$D_\pm^{(l)}(*) = \partial_\pm(*) + \left[A_\pm^{(l)}, *\right], \qquad D_\pm^{(r)}(*) = \partial_\pm(*) + \left[A_\pm^{(r)}, *\right],$$

where $A_\pm^{(l)} = u^{-1}\partial_\pm u$, $A_\pm^{(r)} = \overline{u}^{-1}\partial_\pm\overline{u}$ and the equations (12) imply the following form for the Lax operators[6]

$$
\begin{aligned}
\mathcal{L}_+ &= \partial_+ + \gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+^{(l)}\gamma + z\psi_+ - z^2\Lambda, \\
\mathcal{L}_- &= \partial_- + A_-^{(r)} + z^{-1}\gamma^{-1}\psi_-\gamma - z^{-2}\gamma^{-1}\Lambda\gamma
\end{aligned}
\tag{16}
$$

together with the constraints

$$A_+^{(r)} = \left(\gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+^{(l)}\gamma\right)^\perp + 2\Lambda\psi_+^2, \qquad A_-^{(l)} = \left(-\partial_-\gamma\gamma^{-1} + \gamma A_-^{(r)}\gamma^{-1}\right)^\perp + 2\Lambda\psi_-^2, \tag{17}$$

where we have used the definitions

$$\psi_\pm = [\Lambda, y_{\mp 1}], \qquad [\Lambda, y_{-2}] = \left(\gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+^{(l)}\gamma\right)^\|, \qquad [\Lambda, y_{+2}] = \left(-\partial_-\gamma\gamma^{-1} + \gamma A_-^{(r)}\gamma^{-1}\right)^\|.$$

The $H_L \times H_R$ gauge transformations are implemented by

$$
\begin{aligned}
u &\to uh_l^{-1}, & \overline{u} &\to \overline{u}h_r^{-1}, & \gamma &\to h_l\gamma h_r^{-1}, & \psi_+ &\to h_r\psi_+ h_r^{-1}, & \psi_- &\to h_l\psi_- h_l^{-1} \\
A_+^{(l)} &\to h_l A_+^{(l)} h_l^{-1} - \partial_+ h_l h_l^{-1}, & A_-^{(r)} &\to h_r A_-^{(r)} h_r^{-1} - \partial_- h_r h_r^{-1}
\end{aligned}
\tag{18}
$$

and the curvature components $F_{+-} = [\mathcal{L}_+, \mathcal{L}_-] = z^{-1}F_{+-}^{(-1)} + F_{+-}^{(0)} + zF_{+-}^{(1)}$ are given by

$$
\begin{aligned}
F_{+-}^{(0)} &= -D_-^{(r)}\left(\gamma^{-1}D_+^{(l)}\gamma\right) + \left[D_+^{(l)}, D_-^{(r)}\right] + [\Lambda, \gamma^{-1}\Lambda\gamma] + [\psi_+, \gamma^{-1}\psi_-\gamma], \\
F_{+-}^{(-1)} &= -D_-^{(r)}\psi_+ - [\Lambda, \gamma^{-1}\psi_-\gamma], \qquad F_{+-}^{(1)} = \gamma^{-1}\left(D_+^{(l)}\psi_- + [\Lambda, \gamma\psi_+\gamma^{-1}]\right)\gamma.
\end{aligned}
\tag{19}
$$

The equations of motion are defined by $F_{+-} = 0$.

An infinite tower of local and non-local, bosonic and fermionic conserved charges q_s of 2d Lorentz spin $s/2$ are hidden in the Lax operators (16) and to extract them we employ the so-called Drinfeld-Sokolov (DS) procedure [21]. For simplicity we will restrict to the on-shell gauge $u = \overline{u} = I$ and introduce the following coordinates for the phase space $\mathcal{P}$

$$\mathcal{P}: \quad (Q, \overline{Q}), \quad Q = (q, \psi), \quad \overline{Q} = (\overline{q}, \overline{\psi}), \quad q \equiv \gamma^{-1}\partial_+\gamma, \quad \overline{q} \equiv -\partial_-\gamma\gamma^{-1}, \quad \psi \equiv \psi_+, \quad \overline{\psi} \equiv \psi_-. \tag{20}$$

The positive/negative spin $s/2$ charges q_{-s}/q_s, $s \in \mathbb{Z}^+$ are obtained from (12) and appear in the expansion of the subtracted monodromy matrix $\mathcal{M}(z)$ around $z = 0$ and $z = \infty$ as follows [16]

$$\mathcal{M}(z) = \exp\left[q_0 + q_1 z + q_2 z^2 + ...\right] = \exp\left[q_{-1}/z + q_{-2}/z^2 + ...\right]. \tag{21}$$

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[6]The relation with the notation of [1] is $\psi_+ = \Psi_R$, $\psi_- = \Psi_L$ and $\Lambda = -T$. The mass scale μ is taken as 1.

The charges q_{-s} are computed from

$$\Phi^{-1}(z)\mathcal{L}_+(z)\Phi(z) = \partial_+ - z^2\Lambda + h_+(z), \qquad \Phi^{-1}(z)\mathcal{L}_-(z)\Phi(z) = \partial_- + h_-(z), \qquad h_\pm(z) \in \widehat{\mathfrak{f}}^\perp_{\leq 0}, \tag{22}$$

while the charges q_s from

$$\widetilde{\Phi}^{-1}(z)\mathcal{L}'_-(z)\widetilde{\Phi}(z) = \partial_- - z^{-2}\Lambda + \widetilde{h}_-(z), \qquad \widetilde{\Phi}^{-1}(z)\mathcal{L}'_+(z)\widetilde{\Phi}(z) = \partial_+ + \widetilde{h}_+(z), \qquad \widetilde{h}_\pm(z) \in \widehat{\mathfrak{f}}^\perp_{\geq 0}. \tag{23}$$

The equations of motion $F_{+-} = 0$ imply the following relations

$$\partial_+ h_-(z) - \partial_- h_+(z) + [h_+(z), h_-(z)] = 0, \qquad \partial_+ \widetilde{h}_-(z) - \partial_- \widetilde{h}_+(z) + \left[\widetilde{h}_+(z), \widetilde{h}_-(z)\right] = 0 \tag{24}$$

and provide conservations laws for the dynamical system. Due to the fact that the current components $h(z)$ and $\widetilde{h}(z)$ are related by the parity transformations $z \to z^{-1}$, $+ \to -$, $\gamma \to \gamma^{-1}$, we need to find only one set of these charges. Another important ingredient of this construction is that we still have the action of an infinite dimensional group of transformations

$$\Phi \to \Phi\eta_-, \qquad \widetilde{\Phi} \to \widetilde{\Phi}\eta_+, \qquad \eta_\pm = \exp\beta_\pm, \qquad \beta_-, \beta_+ \in \widehat{\mathfrak{f}}^\perp_{<0}, \widehat{\mathfrak{f}}^\perp_{>0}, \tag{25}$$

that does not change the lhs of (22) and (23) but changes the form of the DS currents in the rhs and this is equivalent to a change in the dressing matrices $\Omega, \widetilde{\Omega}$ in (11). The change in the currents induced by (25) is

$$h'_\pm = \eta_-^{-1} h_\pm \eta_- + \eta_-^{-1}\partial_\pm \eta_-, \qquad \widetilde{h}'_\pm = \eta_+^{-1}\widetilde{h}_\pm \eta_+ + \eta_+^{-1}\partial_\pm \eta_+. \tag{26}$$

We are interested in finding the Poisson superalgebra for the conserved charges associated to the symmetry superalgebra $\mathfrak{s}$, see (15) and for this reason we only need to decompose (24) at Q_H grades $0, \pm 1, \pm 2$ to find $q_{l/r}$, $q_{\pm 1}$ and $q_{\pm 2}$. The answer is [15],[16]

$$q_r = q^\perp + 2\Lambda\psi^2 = 0, \quad q_{-1} = \int_{-\infty}^{+\infty} dx \left(\left[\widetilde{\Lambda}q, \psi\right] - \left(\gamma^{-1}\overline{\psi}\gamma\right)^\perp\right), \quad Str\left(\Lambda, q_{-2}\right) = \int_{-\infty}^{+\infty} dx\,(T_{++} + T_{-+}), \quad (27)$$

$$q_l = \overline{q}^\perp + 2\Lambda\overline{\psi}^2 = 0, \quad q_{+1} = \int_{-\infty}^{+\infty} dx \left(\left[\widetilde{\Lambda}\overline{q}, \overline{\psi}\right] - \left(\gamma\psi\gamma^{-1}\right)^\perp\right), \quad Str\left(\Lambda, q_{+2}\right) = \int_{-\infty}^{+\infty} dx\,(T_{--} + T_{+-}),$$

where we have used the notation $\widetilde{\Lambda} = ad\Lambda$ and the definitions (20). The explicit form for the components $T_{\mu\nu}$ will be written below. One important comment concerning the first equations $q_{l/r}$ is in order. At grade zero, the equations (24) are $\partial_- q_r = \partial_+ q_l = 0$, implying that the quantities $q_r = q_r(x^+)$ and $q_l = q_l(x^-)$ are chiral, but these are precisely the constraints (17) in the gauge[7] $u = \overline{u} = I$. Thus, they vanish and they are not the true gauge charges. However, this constraints have important consequences not only in dictating the explicit form of $q_{\pm 1}, q_{\pm 2}$ but also in the geometric interpretation of $q_{\pm 1}, q_{\pm 2}$ as moment maps, see section 6 below. The true gauge charge is the q_0 appearing in (21) and it cannot be found by the DS procedure, it is of kink-type and is given [16] by $\exp q_0 = \gamma(\infty)^{-1}\gamma(-\infty)$.

In what follows we split the elements $a \in \widehat{\mathfrak{f}}^\perp$ into two groups, $b \in \widehat{\mathfrak{f}}^\perp_{\geq 0}$ and $\overline{b} \in \widehat{\mathfrak{f}}^\perp_{\leq 0}$ in order to separate the two infinite sets of symmetry flows. The equations (12) allows us to write (16) in the following form

$$\mathcal{L}_+ = \partial_+ - A(z^2\Lambda)_{\geq 0}, \qquad \mathcal{L}_- = \partial_- - \gamma^{-1}\widetilde{A}(z^{-2}\Lambda)_{<0}\gamma, \qquad A(b) \equiv \Phi b\Phi^{-1}, \qquad \widetilde{A}(\overline{b}) \equiv \widetilde{\Phi}\overline{b}\widetilde{\Phi}^{-1}, \tag{28}$$

which are manifestly invariant under the action of (25). We can also show that

$$\mathcal{L}_b = \partial_{t_b} + \left(\chi b\chi^{-1}\right)_{\geq 0}, \qquad \mathcal{L}_{\overline{b}} = \gamma^{-1}\left(\partial_{t_{\overline{b}}} + \left(\widetilde{\chi}\overline{b}\widetilde{\chi}^{-1}\right)_{\leq 0}\right)\gamma. \tag{29}$$

---

[7]In this gauge the symmetry $H_L \times H_R$ is reduced from (18) to the chiral Kac-Moody symmetry of the fermionic extension of perturbed WZNW model.

To end this section we recall how to compute the differentials for the current functionals $h(z)$ and $\widetilde{h}(z)$ on the co-adjoint orbits $\mathcal{L}_\pm$ and $\mathcal{L}'_\pm$. For simplicity, we consider h_+ only.

Now that we have introduced the space-time coordinates $x^\pm$, define the following integrated inner product

$$(X,Y)^\pm \equiv \int_{-\infty}^{+\infty} dx^\pm \langle X,Y \rangle, \tag{30}$$

which we will used extensively in what follows[8].

Consider the Hamiltonian $H_b \equiv (b, h_+(z))$ associated to the flow ∂_{t_b}, for some element $b \in \widehat{\mathfrak{f}}^\perp$ of positive Q_H grade. The differentials are defined through the usual relation

$$\frac{d}{d\varepsilon} H_b \left[\mathcal{L}_+ + \varepsilon r_+ \right] \mid_{\varepsilon=0} \equiv (d_Q H_b, r_+), \qquad r_+ = cte \in \widehat{\mathfrak{f}}_{\geq 0}, \qquad d_Q H_b \in \widehat{\mathfrak{f}} \bmod \widehat{\mathfrak{f}}_{>0}.$$

Explicitly, we have that

$$\frac{d}{d\varepsilon} H_b \left[\mathcal{L}_+ + \varepsilon r_+ \right] \mid_{\varepsilon=0} = \left(b, \frac{d}{d\varepsilon} \mathfrak{L}_+^\varepsilon \right) \mid_{\varepsilon=0} = \left(\Phi b \Phi^{-1}, r_+ \right) + (b, [\mathfrak{L}_+, T_y]), \tag{31}$$

where $\mathfrak{L}_+^\varepsilon \equiv \Phi^{-1}(\varepsilon) \mathcal{L}_+(\varepsilon) \Phi(\varepsilon)$, $\mathfrak{L}_+ \equiv \mathfrak{L}_+^{\varepsilon=0}$, $T_y \equiv \Phi^{-1} \widehat{y}(z) \Phi$ and $\widehat{y}(z) \equiv \frac{d}{d\varepsilon} y(\varepsilon) \mid_{\varepsilon=0}$. When the second term in the rhs of (31) vanishes, which is valid for $b \in \mathfrak{z}$ and $b = z\epsilon$, $\epsilon \in \mathfrak{f}_1^\perp$, the differential of H_b is given by

$$d_Q H_b = A(b)_{\leq 0} = d_q H_b + z^{-1} d_\psi H_b,$$

where we have defined

$$d_q H_b = \frac{\delta H_b}{\delta q} \equiv A(b)_0, \qquad d_\psi H_b = \frac{\delta H_b}{\delta \psi} \equiv z A(b)_{-1}. \tag{32}$$

Similar results holds also for h_- and $\widetilde{h}_\pm$ and will be written later. As shown above, the Lax operators $\mathcal{L}_\pm$ are invariant under the action of (25) and its effect on the differentials $d_Q H_b$ is simply a conjugation $b \to \eta_- b \eta_-^{-1}$.

3 Recursion relations and the SSSSG Poisson Structures.

In this section we show how construct recursively an infinite number of Poisson structures for the SSSSG integrable hierarchy[9]. It turns out that due to the twisted nature of the superalgebra (5), all Poisson structures are non-local except one which is precisely the canonical structure associated to the fermionic extension of the WZNW model having (19) as equations of motion (in the gauge $u = \overline{u} = I$). For simplicity, we will consider only the positive symmetry flows $\partial_{t_b} \mathcal{L}_+$. The analysis for any other combination follows exactly the same lines as a consequence of (28) and (29).

Taking $b \in \widehat{\mathfrak{f}}_{\geq 0}^\perp$ and noting that the spectral parameter z has Q_H grade $+1$, we have from (28) the following recursion relations

$$A(b)_{n-1} = z^{-1} A(zb)_n, \qquad A(b)_{n-2} = z^{-2} A(z^2 b)_n, \qquad A(b)_{n-4} = z^{-4} A(z^4 b)_n, \tag{33}$$

where we have used the first relation twice and fourth times in order to get the second and third relations. When the underlying affine algebra $\widehat{\mathfrak{f}}$ is untwisted we have that $z\widehat{\mathfrak{f}} \simeq \widehat{\mathfrak{f}}$ and $b, zb \in \widehat{\mathfrak{f}}$. In this case the first relation leads to the well known local first Poisson structure of the KdV hierarchy and for this reason will not be considered here any further. For the details of its construction in the bosonic limit, the reader is refered to [22]. The second relation is to be used

[8]When it is clear from the context, we will drop the $\pm$ signs in the $(X,Y)^\pm$ integrations.

[9]In [15], this hierarchy was named extended homogeneous integrable hierarchy. A better name would be mKdV/SSSSG or SSSSG for short.

when $\mathfrak{f}_0 = \mathfrak{f}_2$ and $\mathfrak{f}_1 = \mathfrak{f}_3$ implying that $z^2 \widehat{\mathfrak{f}} \simeq \widehat{\mathfrak{f}}$, corresponding to the GS superstring in $AdS_3 \times S^3$. The third relation is to be used when $z^4 \widehat{\mathfrak{f}} \simeq \widehat{\mathfrak{f}}$, corresponding to the GS superstring in $AdS_n \times S^n$ with $n = 2, 5$.

We now proceed to construct the non-local Poisson structures associated to the twisted superalgebra $\widehat{\mathfrak{f}}$, i.e (5).

3.1 Second Poisson structure.

The second structure is canonical, it is the most natural and it is the same in all cases, then we consider it first. Taking an element $b \in \mathfrak{z}$ of positive Q_H grade and dressing the trivial relation $[b, \partial_+ - z^2 \Lambda] = 0$, we have the following compatibility relation

$$[\mathcal{L}_+, A(b)] = 0 \tag{34}$$

allowing to represent the flow $\partial_{t_b} \mathcal{L}_+$, using (29), in two equivalent forms

$$\frac{\partial \mathcal{L}_+}{\partial t_b} = -[A(b)_{\geq 0}, \mathcal{L}_+] = [A(b)_{<0}, \mathcal{L}_+].$$

The first form is equivalent to

$$\frac{\partial q}{\partial t_b} = D_+ d_q H_b, \qquad z \frac{\partial \psi}{\partial t_b} = D_+ A(b)_1 + z \widetilde{\psi} d_q H_b, \tag{35}$$

where we have denoted $\widetilde{\psi} = ad\psi$ and defined $D_+(*) = [\partial_+ + q, *]$. The recursion relation induced by (34) is

$$D_+ A(b)_n = -z \widetilde{\psi} A(b)_{n-1} + z^2 \widetilde{\Lambda} A(b)_{n-2} \tag{36}$$

and can be written in two different but equivalent ways

$$\begin{aligned} A(b)_n &= z u A(b)_{n-1} + z^2 v A(b)_{n-2}, & u &\equiv -D_+^{-1} \widetilde{\psi}, & v &\equiv D_+^{-1} \widetilde{\Lambda}, \\ A(b)_n &= z^{-1} w A(b)_{n+1} + z^{-2} y A(b)_{n+2}, & w &\equiv \widetilde{\Lambda}^{-1} \widetilde{\psi}, & y &\equiv \widetilde{\Lambda}^{-1} D_+. \end{aligned} \tag{37}$$

Using (36) and (32) with $n = 1$, we have

$$D_+ A(b)_1 = -z \widetilde{\psi} d_q H_b + z \widetilde{\Lambda} d_\psi H_b$$

and inserting this result in the second equation in (35) we have[10]

$$\frac{\partial Q}{\partial t_b} = \Theta^{(0)} d_Q H_b, \qquad \Theta^{(0)} \equiv \begin{pmatrix} D_+ & 0 \\ 0 & \widetilde{\Lambda} \end{pmatrix}, \qquad d_Q H_b = \begin{pmatrix} d_q H_b \\ d_\psi H_b \end{pmatrix} \tag{38}$$

where Q was defined in (20). The flow of any functional $\varphi(Q)$ is given by

$$\frac{\partial \varphi(Q)}{\partial t_b} = \left(d_q \varphi, \frac{\partial q}{\partial t_b} \right) + \left(d_\psi \varphi, \frac{\partial \psi}{\partial t_b} \right) = (d_q \varphi, D_+ d_q H_b) + \left(d_\psi \varphi, \widetilde{\Lambda} d_\psi H_b \right)$$

and motivates the following definition for the second Poisson bracket

$$\{\varphi, \psi\}_2 \equiv \left(d_Q \varphi, \Theta^{(0)} d_Q \psi \right) = \int_{-\infty}^{+\infty} dx^+ Str \left(\frac{\delta \varphi}{\delta Q}, \Theta^{(0)} \frac{\delta \psi}{\delta Q} \right) \tag{39}$$

in terms of which

$$\frac{\partial \varphi}{\partial t_b} = \{\varphi, H_b\}_2.$$

---

[10]In what follows we will denote row and column vectors by the same letter.

This is the boost-invariant Poisson structure already considered in [15]. Note that in the whole derivation we did not used the relations (33) at all, hence the second bracket (39) is the same for the two possible situations of interest, cf (9).

To compute the Poisson superalgebra corresponding to $\mathfrak{s}$ in (15) we will need the light-cone components of the conserved currents $h(z)$, $\widetilde{h}(z)$ associated to the elements in $\mathfrak{s}$. They are $b = z\epsilon$, $b = z^2\Lambda$, with $\epsilon \in \mathfrak{f}_1^\perp$ and $\overline{b} = z^{-1}\overline{\epsilon}$, $\overline{b} = z^{-2}\Lambda$ with $\overline{\epsilon} \in \mathfrak{f}_3^\perp$. The DS current components take the form

$$j_+(b) \equiv \langle A(b), \mathcal{L}_+ \rangle, \qquad j_-(b) \equiv -\langle A(b), \mathcal{L}_- \rangle, \qquad \overline{j}_+(\overline{b}) \equiv -\left\langle \widetilde{A}(\overline{b}), \mathcal{L}'_+ \right\rangle, \qquad \overline{j}_-(\overline{b}) \equiv \left\langle \widetilde{A}(\overline{b}), \mathcal{L}'_- \right\rangle \qquad (40)$$

and their associated differentials[11] are

$$
\begin{aligned}
d_+ j_+(b) &= A(b)_{\leq 0}, & d'_+ j_+(b) &= \gamma d_+ j_+(b) \gamma^{-1}, \\
d_- j_-(b) &= -A(b)_{\geq 0}, & d'_- j_-(b) &= \gamma d_- j_-(b) \gamma^{-1}, \\
d'_+ \overline{j}_+(\overline{b}) &= -\widetilde{A}(\overline{b})_{\leq 0}, & d_+ \overline{j}_+(\overline{b}) &= \gamma^{-1} d'_+ \overline{j}_+(\overline{b}) \gamma, \\
d'_- \overline{j}_-(\overline{b}) &= \widetilde{A}(\overline{b})_{\geq 0}, & d_- \overline{j}_-(\overline{b}) &= \gamma^{-1} d'_- \overline{j}_-(\overline{b}) \gamma.
\end{aligned}
\qquad (41)
$$

The notation $d_\pm$ and $d'_\pm$ helps to keep track the domains of definitions in which the current components are defined.

The light-cone second brackets in the Kostant-Kirillov form are [15]

$$
\begin{aligned}
\{\varphi, \psi\}_2(\mathcal{L}_+) &= -\left(\mathcal{L}_+, [d_+\varphi, d_+\psi]_{R_-}\right), & \{\varphi, \psi\}_2(\mathcal{L}_-) &= -\left(\mathcal{L}_-, \gamma^{-1}\left[d'_-\varphi, d'_-\psi\right]_{R_+}\gamma\right), \\
\{\varphi, \psi\}_2(\mathcal{L}'_+) &= -\left(\mathcal{L}'_+, \gamma\left[d_+\varphi, d_+\psi\right]_{R_-}\gamma^{-1}\right), & \{\varphi, \psi\}_2(\mathcal{L}'_-) &= -\left(\mathcal{L}'_-, \left[d'_-\varphi, d'_-\psi\right]_{R_+}\right),
\end{aligned}
\qquad (42)
$$

where $R_- \equiv (\pi_0 - \pi_{<0})/2$ and $R_+ \equiv (\pi_0 - \pi_{>0})/2$ are the ususal R-matrices defined in terms of projectors π. The Poisson bracket on the spatial orbit $\mathcal{L}_x = \mathcal{L}_+ - \mathcal{L}_-$ is defined by

$$\{\varphi, \psi\}_2(\mathcal{L}_x) = \{\varphi, \psi\}_2(\mathcal{L}_+) - \{\varphi, \psi\}_2(\mathcal{L}_-) \qquad (43)$$

and a similar definition holds for $\{\varphi, \psi\}_2(\mathcal{L}'_x)$. All these brackets have to be restricted to the level sets[12] $q_{l/r} = 0$ consequence of (27)

$$q^\perp + 2\Lambda\psi^2 = 0, \qquad \overline{q}^\perp + 2\Lambda\overline{\psi}^2 = 0. \qquad (44)$$

3.2 First Poisson structures.

The evolution equations (35) can be written in two different but completely equivalent ways. By using the two equations of (37) in (35), we have

$$\frac{\partial Q}{\partial t_b} = \left(\begin{array}{c} D_+ u \left(zA(b)_{-1}\right) + D_+ v \left(z^2 A(b)_{-2}\right) \\ \widetilde{\Lambda} u \left(z^2 A(b)_{-2}\right) + \widetilde{\Lambda} v \left(z^3 A(b)_{-3}\right) \end{array} \right)_{\Theta^{(1)}} = \left(\begin{array}{c} D_+ w \left(z^{-1}A(b)_1\right) + D_+ y \left(z^{-2}A(b)_2\right) \\ \widetilde{\Lambda} w \left(A(b)_0\right) + \widetilde{\Lambda} y \left(z^{-1}A(b)_1\right) \end{array} \right)_{\Theta^{(-1)}}. \qquad (45)$$

Now we need to rewrite these expressions in terms of differential forms and to do it this time we need to take into account the recursion relations (33), thus the first Poisson structures are sensitive to the degree of twisting in (9) and in turn this is reflected in the degree of non-locality of the Poisson brackets.

---

[11] For the differentials we have only two terms in the expansions, $(*)_{\leq 0} = (*)_0 + (*)_{-1}$ and $(*)_{\geq 0} = (*)_0 + (*)_{+1}$.

[12] These conditions are automatically satisfied by the soliton solutions, see [16].

Starting with the second relation in (33) which is relevant in the $AdS_3 \times S^3$ case, we get the following expressions for $\Theta^{(1)}$

$$A(b)_{-2} = z^{-2} d_q H_{z^2 b}, \qquad A(b)_{-3} = z^{-3} d_\psi H_{z^2 b}, \qquad zA(b)_{-1} = u d_q H_{z^2 b} + v d_\psi H_{z^2 b}$$

and for $\Theta^{(-1)}$

$$A(b)_1 = z d_\psi H_{z^{-2} b}, \qquad A(b)_2 = z^2 d_q H_{z^{-2} b}, \qquad A(b)_0 = y d_q H_{z^{-2} b} + w d_\psi H_{z^{-2} b}.$$

Inserting these expressions in (45), the first forms become

$$\frac{\partial Q}{\partial t_b} = \Theta^{(\pm 1)} d_Q H_{z^{\pm 2} b}, \qquad \Theta^{(1)} \equiv \begin{pmatrix} D_+ \left(u^2 + v \right) & D_+ uv \\ \widetilde{\Lambda} u & \widetilde{\Lambda} v \end{pmatrix}, \qquad \Theta^{(-1)} \equiv \begin{pmatrix} D_+ y & D_+ w \\ \widetilde{\Lambda} wy & \widetilde{\Lambda} \left(w^2 + y \right) \end{pmatrix} \tag{46}$$

and we can explicitly verify that the three Poisson structures $\Theta^{(\pm 1)}, \Theta^{(0)}$ satisfy the relation

$$\Theta^{(-1)} = \Theta^{(0)} \Theta^{(1)-1} \Theta^{(0)}$$

where

$$\Theta^{(1)-1} = \begin{pmatrix} y & wy \\ wy & \left(w^2 y + y^2 \right) \end{pmatrix} D_+^{-1}$$

and this means that they are all compatible.

Now we consider the third relation in (33) which is relevant to the cases $AdS_n \times S^n$, $n = 2, 5$. We get the following expressions for $\Theta^{(1)}$

$$
\begin{aligned}
A(b)_{-4} &= z^{-4} d_q H_{z^4 b}, \qquad A(b)_{-5} = z^{-5} d_\psi H_{z^4 b}, \\
zA(b)_{-1} &= \left(u^3 + uv + vu \right) d_q H_{z^4 b} + \left(u^2 v + v^2 \right) d_\psi H_{z^4 b}, \\
z^2 A(b)_{-2} &= \left(u^2 + v \right) d_q H_{z^4 b} + uv d_\psi H_{z^4 b}, \\
z^3 A(b)_{-3} &= u d_q H_{z^4 b} + v d_\psi H_{z^4 b}.
\end{aligned}
$$

and for $\Theta^{(-1)}$

$$
\begin{aligned}
A(b)_3 &= z^3 d_\psi H_{z^{-4} b}, \qquad A(b)_4 = z^4 d_q H_{z^{-4} b}, \\
A(b)_0 &= \left(y^2 + w^2 y \right) d_q H_{z^{-4} b} + \left(w^3 + wy + yw \right) d_\psi H_{z^{-4} b}, \\
z^{-1} A(b)_1 &= wy d_q H_{z^{-4} b} + \left(w^2 + y \right) d_\psi H_{z^{-4} b}, \\
z^{-2} A(b)_2 &= y d_q H_{z^{-4} b} + w d_\psi H_{z^{-4} b}.
\end{aligned}
$$

Inserting this expression in (45) above we have

$$
\begin{aligned}
\frac{\partial Q}{\partial t_b} &= \Theta^{(\pm 1)} d_Q H_{z^{\pm 4} b}, \\
\Theta^{(1)} &\equiv \begin{pmatrix} D_+ \left(u^4 + u^2 v + uvu + vu^2 + v^2 \right) & D_+ \left(u^3 v + uv^2 + vuv \right) \\ \widetilde{\Lambda} \left(u^3 + uv + vu \right) & \widetilde{\Lambda} \left(u^2 v + v^2 \right) \end{pmatrix}, \tag{47} \\
\Theta^{(-1)} &\equiv \begin{pmatrix} D_+ \left(w^2 y + y^2 \right) & D_+ \left(w^3 + wy + yw \right) \\ \widetilde{\Lambda} \left(wy^2 + w^3 y + ywy \right) & \widetilde{\Lambda} \left(w^4 + w^2 y + wyw + yw^2 + y^2 \right) \end{pmatrix} \tag{48}
\end{aligned}
$$

and in a similar way we can verify the compatibility relation

$$\Theta^{(-1)} = \Theta^{(0)} \Theta^{(1)-1} \Theta^{(0)}$$

11

with

$$\Theta^{(1)-1} = \begin{pmatrix} w^2y + y^2 & w^3y + wy^2 + ywy \\ wy^2 + w^3y + ywy & w^4y + w^2y^2 + (wy)^2 + yw^2y + y^3 \end{pmatrix} D_+^{-1}.$$

The flow of any functional $\varphi(Q)$ is given by

$$\frac{\partial \varphi(Q)}{\partial t_b} = \left(d_Q\varphi, \frac{\partial Q}{\partial t_b} \right) = \left(d_Q\varphi, \Theta^{(\pm 1)}d_Q H_{z^{\pm 4}b} \right)$$

and motivates the following definition for the first Poisson brackets

$$\{\varphi, \psi\}_{\pm 1} \equiv \left(d_Q\varphi, \ \Theta^{(\pm 1)}d_Q\psi \right) = \int_{-\infty}^{+\infty} dx^+ Str \left(\frac{\delta\varphi}{\delta Q}, \Theta^{(\pm 1)} \frac{\delta\psi}{\delta Q} \right)$$

in terms of which

$$\frac{\partial \varphi}{\partial t_b} = \{\varphi, H_{z^{\pm 4}b}\}_{\pm 1} \, .$$

Then, we have shown that

$$\{\varphi, H_b\}_2 = \{\varphi, H_{z^{\pm 4}b}\}_{\pm 1} \, . \tag{49}$$

In this way we can construct recursively an infinite family of non-local Poisson structures for the SSSSG integrable hierarchy governing the Pohlmeyer reduced models

$$\frac{\partial Q}{\partial t_b} = \Theta^{(\pm n)}d_Q H_{z^{\pm 4n}b}, \qquad n \in \mathbb{Z}^+.$$

The computation for the higher Poisson bi-vectors $\Theta^{(\pm n)}$, $n \geq 2$ becomes rather cumbersome. However, in the bosonic limit we can write $\Theta^{(\pm n)}$ in closed form

$$\Theta^{(n)} = \left(\widetilde{\Lambda} D_+^{-1} \widetilde{\Lambda} D_+^{-1} \right)^n \Theta^{(0)}, \qquad \Theta^{(0)} = D_+, \qquad n \in \mathbb{Z}.$$

The degree of non-locality n and the behavior of the brackets under Lorentz boosts $x^\pm \to \lambda^{\pm 1}x^\pm$ are correlated and the only boost invariant bracket is $\Theta^{(0)}$. It is very important to recall that all Poisson structures have to be restricted, in field space, to the slices $q_{l/r} = 0$.

As we will see in the next section, the special combination

$$\Theta_\sigma = \Theta^{(-1)} - 2\Theta^{(0)} + \Theta^{(+1)} \tag{50}$$

corresponds to the canonical Poisson structure of the GS superstring σ-model after gauge fixing all the local symmetries. From this we conclude that the Pohlmeyer reduced model described by $\mathcal{L}_\pm$ and supplemented by the constraints $q_{l/r} = 0$ carry all the classical information of the σ-model that generated it.

4 Connection with the GS superstring sigma model.

In this section we rewrite some known results of the GS superstring σ-model and show how they fit in the SSSSG integrable hierarchy approach. In particular, the relation between the canonical Poisson structure of the σ-model and the Poisson structures constructed from the recursion operators, the relation between commuting charges in terms of the Zukhovsky map and the relation between Lax pair representations.

4.1 Relation between Poisson structures.

Due to the relevance of the result of [23] in relation to ours, here we briefly review it. The goal is to find the explicit form of the canonical Poisson bi-vector Θ_σ when restricted to the symplectic leaves left after fixing all the local gauge symmetries of the σ-model.

The GSsσ model is defined by the following action functional

$$S_{GS} = \frac{1}{2} \int_\Sigma Str\left(J^{(2)} \wedge *J^{(2)} + \kappa J^{(1)} \wedge J^{(3)} \right), \qquad J \equiv f^{-1} df, \tag{51}$$

where Σ denotes the string world-sheet. In the conformal gauge[13] and in the partially fixed kappa symmetry gauge [1] $J_+^{(3)} = J_-^{(1)} = 0$, the canonical symplectic form of (51) constructed in [23] takes the form $\Omega_\sigma = \Omega_\sigma^+ - \Omega_\sigma^-$, with[14]

$$\Omega_\sigma^+ = \frac{1}{2}\left(\left(f^{-1}\delta f\right)^{(2)}, \wedge D_+^{(0)} \left(f^{-1}\delta f\right)^{(2)} + 2J_+^{(2)} \left(f^{-1}\delta f\right)^{(1)}, \wedge \left(f^{-1}\delta f\right)^{(1)} \right), \tag{52}$$

and $\Omega_\sigma^- = \Omega_\sigma^+(+ \leftrightarrow -, 1 \leftrightarrow 3)$.

The idea is to find the inverse $\Omega_\sigma^{-1} = \Theta_\sigma$ by using Hamiltonian vectors fields. Consider a left-invariant vector field $X_\xi = f\xi$ in which $f \in F = \exp\mathfrak{f}$, $\mathfrak{f}$ is defined by (3) and ξ is the vector field at the supergroup identity. This vector field satisfy $\left(f^{-1}\delta f\right)(X_\xi) = \xi$ and as consequence we have that

$$\Omega_\sigma^+\left(X_\eta, X_\xi\right) = \frac{1}{2}\left(\eta^{(2)}, \overleftrightarrow{D_+^{(0)}}\xi^{(2)} - \eta^{(1)}, \overrightarrow{a_{2+}}\xi^{(1)} \right), \tag{53}$$

where $X \overleftrightarrow{\mathcal{O}} Y \equiv X\left(\mathcal{O}Y\right) - \left(\mathcal{O}Y\right)X$ and $a_{i+} = ad_{J_+^{(i)}}$. Considering X_η, X_ξ as the Hamiltonian vector fields generated by the functionals F, G, we have

$$\{F,G\}_\sigma = \Omega_\sigma^+\left(X_\eta, X_\xi\right) = X_\eta\left(G\right) = \delta G\left(X_\eta\right) = -X_\xi\left(F\right) = -\delta F\left(X_\xi\right).$$

To find the components η of the Hamiltonian vector field X_η, we have to solve the relation

$$\Omega_\sigma\left(X_\eta, X_\xi\right) = -\delta F\left(X_\xi\right), \qquad \forall \xi. \tag{54}$$

There is a freedom in choosing the component $\xi^{(0)}$ and in the following we will take it as zero. Using the relation $\delta J_\mu = D_\mu\left(f^{-1}\delta f\right)$ we get

$$\delta J_+^{(3)}\left(X_\xi\right) = 0 = D_+^{(0)}\xi^{(3)} + a_{1+}\xi^{(2)} + a_{2+}\xi^{(1)}$$

because we need to maintain the gauge $J_+^{(3)} = 0$ and this allows to determine one of the components, in this case we take

$$\xi^{(3)} = -D_+^{(0)-1}\left(a_{1+}\xi^{(2)} + a_{2+}\xi^{(1)}\right). \tag{55}$$

By replacing this component on the other contractions we have

$$
\begin{aligned}
\delta J_+^{(2)}\left(X_\xi\right) &= a_{1+}\xi^{(1)} + D_+^{(0)}\xi^{(2)}, \\
\delta J_+^{(1)}\left(X_\xi\right) &= \left(D_+^{(0)} - a_{2+}D_+^{(0)-1}a_{2+}\right)\xi^{(1)} - a_{2+}D_+^{(0)-1}a_{1+}\xi^{(2)}, \\
\delta J_+^{(0)}\left(X_\xi\right) &= -a_{1+}D_+^{(0)-1}a_{2+}\xi^{(1)} + \left(a_{2+} - a_{1+}D_+^{(0)-1}a_{1+}\right)\xi^{(2)}.
\end{aligned}
\tag{56}
$$

---

[13] The conventions used in conformal gauge are $\epsilon^{+-} = -1$, $\eta_{+-} = 1$, $\kappa = -1$, $(*J)_\alpha = J_\rho \epsilon^{\rho\lambda}\eta_{\lambda\alpha}$. The covariant derivative is $D \equiv d + ad_J$.

[14] As used before, we write $(X,Y)^\pm \equiv \int dx^\pm Str(X,Y)$ and omit the $\pm$ indices when there is not ambiguity.

With this in mind we can compute the rhs of (54) by using (56) and the obvious relation

$$\delta F\left(X_\xi\right) = \left(\frac{\delta F}{\delta J_+^{(a)}}, \delta J_+^{(a)}\left(X_\xi\right)\right).$$

The answer is

$$\left(\frac{\delta F}{\delta J_+^{(0)}}, \delta J_+^{(0)}\left(X_\xi\right)\right) = \left(a_{2+}D_+^{(0)-1}a_{1+}\frac{\delta F}{\delta J_+^{(0)}}, \xi^{(1)} + \left(-a_{2+} + a_{1+}D_+^{(0)-1}a_{1+}\right)\frac{\delta F}{\delta J_+^{(0)}}, \xi^{(2)}\right),$$

$$\left(\frac{\delta F}{\delta J_+^{(1)}}, \delta J_+^{(1)}\left(X_\xi\right)\right) = \left(\left(-D_+^{(0)} + a_{2+}D_+^{(0)-1}a_{2+}\right)\frac{\delta F}{\delta J_+^{(1)}}, \xi^{(1)} + a_{1+}D_+^{(0)-1}a_{2+}\frac{\delta F}{\delta J_+^{(1)}}, \xi^{(2)}\right),$$

$$\left(\frac{\delta F}{\delta J_+^{(2)}}, \delta J_+^{(2)}\left(X_\xi\right)\right) = \left(-a_{1+}\frac{\delta F}{\delta J_+^{(2)}}, \xi^{(1)} - D_+^{(0)}\frac{\delta F}{\delta J_+^{(2)}}, \xi^{(2)}\right).$$

The lhs of (54) was already computed in (53) and it is given by

$$\Omega_\sigma\left(X_\eta, X_\xi\right) = \left(a_{2+}\eta^{(1)}, \xi^{(1)} - D_+^{(0)}\eta^{(2)}, \xi^{(2)}\right).$$

By equating both sides, we determine all the components of the vector field X_η, $\eta^{(3)}$ is determined by (55),

$$\eta^{(1)} = -D_+^{(0)-1}a_{1+}\frac{\delta F}{\delta J_+^{(0)}} + \left(a_{2+}^{-1}D_+^{(0)} - D_+^{(0)-1}a_{2+}\right)\frac{\delta F}{\delta J_+^{(1)}} + a_{2+}^{-1}a_{1+}\frac{\delta F}{\delta J_+^{(2)}}, \tag{57}$$

$$\eta^{(2)} = -\left(D_+^{(0)-1}a_{2+} - \left(D_+^{(0)-1}a_{1+}\right)^2\right)\frac{\delta F}{\delta J_+^{(0)}} + D_+^{(0)-1}a_{1+}D_+^{(0)-1}a_{2+}\frac{\delta F}{\delta J_+^{(1)}} - \frac{\delta F}{\delta J_+^{(2)}}$$

and this is enough to find the inverse of the symplectic form.

Once the Hamiltonian vector field X_η of F is found, we easily obtain the Poisson brackets of the functional F with the currents J by taking $G = J_+^{(b)}$, i.e,

$$\left\{F, J_+^{(b)}\right\}_\sigma = \delta J_+^{(b)}\left(X_\eta\right).$$

The components of the Poisson bi-vector $\Theta_\sigma = \Omega_\sigma^{-1}$ are, by definition, given by the Poisson brackets of the phase space coordinates J among themselves

$$\left\{J_+^{(a)}, J_+^{(b)}\right\}_\sigma \equiv [\Theta_\sigma]_{ab}.$$

Replacing the components (57) and specializing to the case $F = J_+^{(a)}$, we obtain

$$\left\{J_+^{(2)}, J_+^{(2)}\right\}_\sigma = D_+^{(0)} - a_{1+}a_{2+}^{-1}a_{1+}, \qquad \left\{J_+^{(2)}, J_+^{(1)}\right\} = -a_{1+}a_{2+}^{-1}D_+^{(0)}, \qquad \left\{J_+^{(2)}, J_+^{(0)}\right\} = a_{2+},$$

$$\left\{J_+^{(1)}, J_+^{(0)}\right\}_\sigma = a_{1+} - a_{2+}D_+^{(0)-1}a_{1+}D_+^{(0)-1}a_{2+} + a_{2+}\left(D_+^{(0)-1}a_{1+}\right)^3 - \left(a_{2+}D_+^{(0)-1}\right)^2 a_{1+},$$

$$\left\{J_+^{(1)}, J_+^{(1)}\right\}_\sigma = -\left(D_+^{(0)} - a_{2+}D_+^{(0)-1}a_{2+}\right)a_{2+}^{-1}\left(D_+^{(0)} - a_{2+}D_+^{(0)-1}a_{2+}\right) + a_{2+}\left(D_+^{(0)-1}a_{1+}\right)^2 D_+^{(0)-1}a_{2+},$$

$$\left\{J_+^{(0)}, J_+^{(0)}\right\}_\sigma = a_{2+}D_+^{(0)-1}a_{2+} - \left(a_{1+}D_+^{(0)-1}\right)^2 a_{2+} - a_{2+}\left(D_+^{(0)-1}a_{1+}\right)^2 -$$

$$-a_{1+}D_+^{(0)-1}a_{2+}D_+^{(0)-1}a_{1+} + a_{1+}\left(D_+^{(0)-1}a_{1+}\right)^3.$$

This brackets can be split according to their behavior under Lorentz boost [23]

$$\left\{J_+^{(2)}, J_+^{(2)}\right\}_\sigma^{[2]} = D_+^{(0)} - a_{1+}a_{2+}^{-1}a_{1+}, \qquad \left\{J_+^{(2)}, J_+^{(1)}\right\}_\sigma^{[2]} = -a_{1+}a_{2+}^{-1}D_+^{(0)}, \qquad \left\{J_+^{(1)}, J_+^{(1)}\right\}_\sigma^{[2]} = -D_+^{(0)}a_{2+}^{-1}D_+^{(0)} \tag{58}$$

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and

$$\left\{J_+^{(2)}, J_+^{(0)}\right\}_\sigma^{[0]} = a_{2+}, \qquad \left\{J_+^{(1)}, J_+^{(0)}\right\}_\sigma^{[0]} = a_{1+}, \qquad \left\{J_+^{(1)}, J_+^{(1)}\right\}_\sigma^{[0]} = 2a_{2+} \tag{59}$$

and

$$\left\{J_+^{(1)}, J_+^{(0)}\right\}_\sigma^{[-2]} = -a_{2+}D_+^{(0)-1}a_{1+}D_+^{(0)-1}a_{2+} + a_{2+}\left(D_+^{(0)-1}a_{1+}\right)^3 - \left(a_{2+}D_+^{(0)-1}\right)^2 a_{1+}, \tag{60}$$

$$\left\{J_+^{(1)}, J_+^{(1)}\right\}_\sigma^{[-2]} = -a_{2+}\left(D_+^{(0)-1}a_{2+}\right)^2 + a_{2+}\left(D_+^{(0)-1}a_{1+}\right)^2 D_+^{(0)-1}a_{2+},$$

$$\left\{J_+^{(0)}, J_+^{(0)}\right\}_\sigma^{[-2]} = a_{2+}D_+^{(0)-1}a_{2+} - \left(a_{1+}D_+^{(0)-1}\right)^2 a_{2+} + a_{2+}\left(D_+^{(0)-1}a_{1+}\right)^2 -$$
$$- a_{1+}D_+^{(0)-1}a_{2+}D_+^{(0)-1}a_{1+} + a_{1+}\left(D_+^{(0)-1}a_{1+}\right)^3.$$

The functionals $\{*, *\}_\sigma^{[\lambda]}$, $\lambda = -2, 0, 2$ define three mutually compatible Poisson brackets providing the bi-Hamiltonian structure of the σ-model in terms of which the Poisson bracket decomposes as follows

$$\{F, G\}_\sigma = \{F, G\}^{[-2]} + \{F, G\}^{[0]} + \{F, G\}^{[2]}. \tag{61}$$

In order to make contact with the results of section 3 we need to write the bracket $\{F, G\}_\sigma$ in terms of the currents $J_+^{(0)}, J_+^{(1)}$ only, which are the fundamental variables we are using for the Pohlmeyer reduced models, see (20). All these brackets make sense only on gauge invariant functionals F, i.e $F\left(J + \delta_{\xi^{(0)}} J\right) = F(J)$ and this condition imply that

$$D_+^{(0)}\frac{\delta F}{\delta J_+^{(0)}} + a_{1+}\frac{\delta F}{\delta J_+^{(1)}} + a_{2+}\frac{\delta F}{\delta J_+^{(2)}} = 0 \rightarrow \frac{\delta F}{\delta J_+^{(2)}} = -a_{2+}^{-1}\left(D_+^{(0)}\frac{\delta F}{\delta J_+^{(0)}} + a_{1+}\frac{\delta F}{\delta J_+^{(1)}}\right), \tag{62}$$

allowing to eliminate the functional derivative $\frac{\delta F}{\delta J_+^{(2)}}$ in all the expressions. The last bracket (60) is already in the desired form and we can write it as

$$\{F, G\}^{[-2]} = \left(\frac{\delta F}{\delta J_+^{(a)}}, \Theta_{ab}^{[-2]}\frac{\delta F}{\delta J_+^{(b)}}\right), \qquad U \equiv -D_+^{(0)-1}a_{1+}, \qquad V \equiv -D_+^{(0)-1}a_{2+},$$

$$\Theta^{[-2]} = \begin{pmatrix} D_+^{(0)}\left(U^4 + U^2V + UVU + VU^2 + V^2\right) & D_+^{(0)}\left(U^3V + UV^2 + VUV\right) \\ -a_{2+}\left(U^3 + UV + VU\right) & -a_{2+}\left(U^2V + V^2\right) \end{pmatrix},$$

where $a, b = 0, 1$. Using (62) in (59) we easily get

$$\{F, G\}^{[0]} = \left(\frac{\delta F}{\delta J_+^{(a)}}, \Theta_{ab}^{[0]}\frac{\delta F}{\delta J_+^{(b)}}\right), \qquad \Theta^{[0]} = -2\begin{pmatrix} D_+^{(0)} & 0 \\ 0 & -a_{2+} \end{pmatrix}$$

and in (58) we get

$$\{F, G\}^{[2]} = \left(\frac{\delta F}{\delta J_+^{(a)}}, \Theta_{ab}^{[2]}\frac{\delta F}{\delta J_+^{(b)}}\right), \qquad W \equiv -a_{2+}^{-1}a_{1+}, \qquad Y \equiv -a_{2+}^{-1}D_+^{(0)},$$

$$\Theta^{[2]} = \begin{pmatrix} D_+^{(0)}\left(W^2Y + Y^2\right) & D_+^{(0)}\left(W^3 + WY + YW\right) \\ -a_{2+}\left(WY^2 + W^3Y + YWY\right) & -a_{2+}\left(W^2 + W^2Y + WYW + YW^2 + Y^2\right) \end{pmatrix}.$$

The full restriction of Ω_σ to the reduced phase space is accomplished by the currents $J_+^{(2)}$ satisfying the Virasoro constraints by the currents $J_+^{(1)}$ satisfying the condition $J_+^{(1)} \in \text{Im } a_{2+}$, fixing the residual kappa symmetry. Both conditions are satisfied by the first set of Pohlmeyer variables

$$J_+^{(0)} = q, \qquad J_+^{(1)} = \psi, \qquad J_+^{(2)} = -\Lambda, \qquad D_+^{(0)} = D_+$$

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and this imply that $U = u$, $V = v$, $W = w$, $Y = y$, where u, v, w, and y were defined in section 3. Comparing with the previous results (38), (47) and (48) we conclude that

$$\Theta^{[-2]} = \Theta^{(1)}, \qquad \Theta^{[0]} = -2\theta^{(0)}, \qquad \Theta^{[2]} = \Theta^{(-1)},$$
$$\Theta_\sigma = \Theta^{[-2]} + \Theta^{[0]} + \Theta^{[2]} = \Theta^{(1)} - 2\Theta^{(0)} + \Theta^{(1)}$$

as announce in (50). To find Θ^σ_- we simply replace $(+ \leftrightarrow -), (1 \leftrightarrow 3)$ in the results above. In this case the second set of Pohlmeyer variables is

$$J_-^{(0)} = 0, \qquad J_-^{(3)} = \gamma^{-1}\overline{\psi}\gamma, \qquad J_-^{(2)} = -\gamma^{-1}\Lambda\gamma, \qquad D_-^{(0)} = \partial_-. \tag{63}$$

Notice that there is a subtlety in the replacement of (63) to get Θ^σ_-. The current $J_-^{(0)}$ is zero on the (-) sector of the light-cone and in principle the functionals $\frac{\delta F}{\delta J_-^{(0)}}$ are not well defined. In order for these results to make sense, we interpret the reduced sigma model light cone phase space as being parametrized by $(Q, \overline{Q})$ with Q on $\mathcal{L}_+$ and $\overline{Q}$ on $\mathcal{L}'_-$, see the definitions (20), with the Poisson brackets defined in section 3.

What is remarkable in the computation of [23] reproduced here, is that the degree of non-locality of the first Poisson structures is determined entirely by the gauge fixing of all the local symmetries of the GS superstring σ-model (51), while in the SSSSG model the degree of non-locality is determined entirely by the twisted nature of the superalgebra $\widehat{\mathfrak{f}}$ through the recursion relation determined by $z^{\pm 4n}\widehat{\mathfrak{f}} \simeq \widehat{\mathfrak{f}}$ and traced back to the $\mathbb{Z}_4$ grading of the semi-symmetric space in which the string propagates. The reduced model although manifestly relativistic carries all the non-relativistic information of the former sigma model it comes from. As mentioned at the end of the last section, the Poisson structures $\Theta^{[0]}, \Theta^{[\pm 2]}$ have to restricted to the level manifolds $q_{l/r} = 0$.

4.2   Relation between commuting charges.

In this section we show how to construct σ- model commuting charges out of the SSSSG commuting charges. In the process we also show the integrable origin of the Zukhovski variable and the need for twisting the inner product in the sigma model side, in the sense of [24].

In the (+) sector of the light-cone phase space, the reduced σ-model and the SSSSG model are parametrized by the same coordinates $Q = (q, \psi)$, the only difference being their Poisson structures. Then, it is natural to try to relate the symmetry flows on both sides. For simplicity we analyze one sector only. We want to construct a sigma model current component $I_+(b)$ from the SSSSG model component $j_+(b)$, both generating the same flow on functionals of the phase space $\mathcal{L}_+$. In other words, we want to solve the following relation

$$\frac{\partial \varphi(\mathcal{L}_+)}{\partial t_b} = \{\varphi, j_+(b)\}_2 (\mathcal{L}_+) = \{\varphi, I_+(b)\}_\sigma (\mathcal{L}_+), \qquad b \in \mathfrak{z},$$

which is equivalent to

$$\Theta^{(0)} d_Q j_+(b) = \Theta_\sigma d_Q I_+(b). \tag{64}$$

To find $I_+(b)$ we make the following superposition ansatz

$$d_Q I_+(b) = d_Q j_+(\phi(z)b), \qquad \phi(z) \equiv \sum_{n \in \mathbb{Z}} c_n z^{4n}, \tag{65}$$

because we need $\phi(z)b \in \widehat{\mathfrak{f}}$ to be in the superalgebra $\widehat{\mathfrak{f}}$, cf (9). The equation (64) can be written as follows

$$\Theta^{(0)} d_Q j_+(b) = \Theta_\sigma d_Q j_+(\phi(z)b) = \Theta^{(0)} d_Q j_+(\phi(z)(z^4 - 2 + z^{-4})b),$$

where we have used the relation (49) connecting the first set of isospectral flows of the SSSSG hierarchy and this determines $\phi(z)$ unambiguously to be[15]

$$\phi(z) = \frac{1}{(z^4 - 2 + z^{-4})} = \frac{1}{(z^2 - z^{-2})^2} = \frac{z^{\pm 4}}{(1 - z^{\pm 4})^2} = \sum_{n=0}^{\infty} n z^{\pm 4n}. \tag{66}$$

In the last term we have expanded around $z = 0$ and $z = +\infty$ and this fixes all the coefficients c_n to be

$$\begin{aligned} \text{around } z &= 0, & c_n &= 0 \text{ for } n \leq 0 & \text{and} & & c_n &= n \quad \text{for } n > 0, \\ \text{around } z &= +\infty, & c_n &= 0 \text{ for } n \geq 0 & \text{and} & & c_n &= -n \quad \text{for } n < 0. \end{aligned}$$

For this result to make sense in the superalgebra $\widehat{\mathfrak{f}}$, we have to consider $\phi(z)$ as a power series in $z^{\pm 4n}$ around $z = 0$ for b of positive Q_H grade with associated $j_+(b)$ and around $z = +\infty$ for $\overline{b}$ of negative Q_H grade with associated $\overline{j}_+(\overline{b})$ and this is because the positive and negative flows are not connected by the recursion relations. Thus, the sigma model current component $I_+(b)$ is expressed as an infinite linear combination of SSSSG components $j_+(b)$ over b. What we have is a linear combination of the $j_+(b)$'s attached to the south pole on the Riemann sphere parametrized by the spectral parameter z. The same happens for $\overline{I}_+(\overline{b})$ and $\overline{j}_+(\overline{b})$ at the north pole. Then, we have

$$I_+(b) = j_+(\phi(z)b), \qquad \overline{I}_+(\overline{b}) = \overline{j}_+(\phi(z)\overline{b}). \tag{67}$$

To include the (-) sector of the light-cone formulation we use (28) and (29) to prove the equivalent recursion relations[16]

$$\begin{aligned} \Theta_-^{(0)} d_{\overline{Q}} j_-(b) &= \Theta_-^{(-1)} d_{\overline{Q}} j_-(z^{-4}b) = \Theta_-^{(1)} d_{\overline{Q}} j_-(z^4 b), \\ \Theta_-^{(0)} d_{\overline{Q}} \overline{j}_-(\overline{b}) &= \Theta_-^{(-1)} d_{\overline{Q}} \overline{j}_-(z^{-4}\overline{b}) = \Theta_-^{(1)} d_{\overline{Q}} \overline{j}_-(z^4\overline{b}), \end{aligned}$$

which can alternatively be obtained by parity. Then, for the negative current components we obtain

$$I_-(b) = j_-(\phi(z)b), \qquad \overline{I}_-(\overline{b}) = \overline{j}_-(\phi(z)\overline{b}). \tag{68}$$

From (27) we define $q(z^2\Lambda) \equiv Str(\Lambda, q_{-2})$ and $\overline{q}(z^{-2}\Lambda) \equiv Str(\Lambda, q_{-2})$, which clearly generates flows along the (x^+, x^-) directions in the flat world-sheet. From these results we see that the σ-model charges generating the same flows on $\mathcal{L}_x$ induced by $q(z^2\Lambda), \overline{q}(z^{-2}\Lambda)$ are

$$\begin{aligned} q_\sigma(z^2\Lambda) &= \int_{-\infty}^{+\infty} dx \left(I_+(z^2\Lambda) + I_-(z^2\Lambda) \right) = \int_{-\infty}^{+\infty} dx \left(j_+(\phi(z)z^2\Lambda) + j_-(\phi(z)z^2\Lambda) \right), \tag{69} \\ \overline{q}_\sigma(z^{-2}\Lambda) &= \int_{-\infty}^{+\infty} dx \left(\overline{I}_+(z^{-2}\Lambda) + \overline{I}_-(z^{-2}\Lambda) \right) = \int_{-\infty}^{+\infty} dx \left(\overline{j}_+(\phi(z)z^{-2}\Lambda) + \overline{j}_-(\phi(z)z^{-2}\Lambda) \right). \end{aligned}$$

This is a nice result because we can associate conserved charges generating (x^+, x^-) translations despite of the fact that the Virasoro constraints have been already imposed $Str(J_\pm^{(2)}, J_\pm^{(2)}) = 0$.

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[15]In the reduction of the σ-model in $AdS_3 \times S^3$, we have $\phi(z) = (z^2 - 2 + z^{-2})$. This decrease in the grade of non-locality, cf (46), in contrast to the $AdS_n \times S^n$, $n = 2, 5$ situation can be understood to be a consequence of the group structure of the target space of the σ-model.

[16]Using $\theta_\pm^{(0)}$ we can rewrite the equations of motion (19), in the gauge $u = \overline{u} = I$, in an elegant compact form

$$\partial_- Q = -\theta_+^{(0)} d_Q \overline{j}_+(z^{-2}\Lambda), \qquad \partial_+ \overline{Q} = -\gamma \theta_-^{(0)} d_{\overline{Q}} j_-(z^2\Lambda)\gamma^{-1}.$$

A closer look to the result (69) suggests that the inner product on sigma model have to be twisted, in the sense of [24]. What we have is the following: from (67), (68) we notice, considering $j_+(b)$ only, that

$$j_+(b) = \int_{-\infty}^{+\infty} dx^+ \left[\oint \frac{dz}{2\pi i} \frac{1}{z} Str \left(\Phi b \Phi^{-1}, \mathcal{L}_+ \right) \right], \qquad I_+(b) = \int_{-\infty}^{+\infty} dx^+ \left[\oint \frac{dz}{2\pi i} \frac{\phi(z)}{z} Str \left(\Phi b \Phi^{-1}, \mathcal{L}_+ \right) \right].$$

From this we naturally identify two different inner products that can be defined on the superalgebra $\widehat{\mathfrak{f}}$, cf (5). The simplest one for the SSSSG model already defined in (8) and a twisted one for the σ−model and defined by, see [24],

$$\langle X, Y \rangle_\phi \equiv \oint \frac{du}{2\pi i} Str \left(X(z), Y(z) \right), \tag{70}$$

where $du = \frac{dz}{z} \phi(z)$, $u = \frac{1}{16} Z$ and $Z = 2\frac{1+z^4}{1-z^4}$ is the Zukhovsky variable and we see how the map SSSSG Model$\rightarrow \sigma$-Model is implemented by the change of variables $z \rightarrow u = u(z)$. Using these products we get, in compact form

$$j_+(b) = (A(b), \mathcal{L}_+), \qquad I_+(b) = (A(b), \mathcal{L}_+)_\phi, \tag{71}$$

where we have taken (30) into account. For $\overline{j}_+(\overline{b})$ and $\overline{I}_-(\overline{b})$ we use the parity transformations. Notice that by writing the evolution equations on the σ-model in terms of $\Theta^{(0)}$ instead of Θ_σ all the non-localities of the Poisson brackets are removed. This also explains the integrable origin of the Zukhovsky variable Z.

Finally, we have shown that

$$\{q(b), q(b')\}_2 (\mathcal{L}_x) = 0 \rightarrow \{q_\sigma(b), q_\sigma(b')\}_\sigma (\mathcal{L}_x) = 0,$$
$$\left\{ \overline{q}(\overline{b}), \overline{q}(\overline{b}') \right\}_2 (\mathcal{L}'_x) = 0 \rightarrow \left\{ \overline{q}_\sigma(\overline{b}), \overline{q}_\sigma(\overline{b}') \right\}_\sigma (\mathcal{L}'_x) = 0,$$

for $b, b' \in \mathfrak{z}$ with positive Q_H grade and $\overline{b}, \overline{b}' \in \mathfrak{z}$ with negative Q_H grade.

One comment is in order, the relation between the charges seems to be valid only for the elements $b, \overline{b} \in \mathfrak{z}$, because for the differentials of the currents $j(b)$, $\overline{j}(\overline{b})$, the second term in the rhs of (31) is absent. In this case the ansatz (65) works fine. It is important to see under what conditions we can apply these results to the elements $b, \overline{b} \in \widehat{\mathfrak{c}}$ because this would allow to define 2d fermionic symmetry flows on the sigma model by maping the fermionic conserved charges of the SSSSG model.

Remark 1 *It would be interested to study the relation between the reduction of the Kostant-Kirillov bracket on the σ-model using the twisted inner product $(*, *)_\phi$ defined in [24], and the non-local Poisson bracket $\{*, *\}_\sigma$ constructed in [23], i.e (61).*

4.3 Relation between Lax representations.

In this section we show the relation between the Lax representation of the σ-model and the SSSSG model. We follow [25] and use the light-cone frame formulation of [1]. The idea is to recall the construction of the Lax pair for the σ-model in an arbitrary world-sheet Σ and then apply the Pohlmeyer reduction to recover the Lax operators defined in (16).

The Lagrangian of the GS σ-model action (51) is[17]

$$L_{GS} = \frac{1}{2} Str \left(\gamma^{\mu\nu} J_\mu^{(2)} J_\nu^{(2)} + \kappa \epsilon^{\mu\nu} J_\mu^{(1)} J_\nu^{(3)} \right) dx^0 \wedge dx^1.$$

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[17]We use $J^{(2)} \wedge *J^{(2)} = \gamma^{\mu\nu} J_\mu^{(2)} J_\nu^{(2)} dx^0 \wedge dx^1$, $\gamma^{\mu\nu} = \sqrt{|h|} h^{\mu\nu}$ and the light-cone conventions introduced in the section 4.1.

Introduce the light-cone frame zweibein e in order to write $h^{\mu\nu} = e^\mu_\alpha e^\nu_\beta \eta^{\alpha\beta}$, $h_{\mu\nu} = e^\alpha_\mu e^\beta_\nu \eta_{\alpha\beta}$, $J_\mu = e^\alpha_\mu J_\alpha$, $J_\alpha = e^\mu_\alpha J_\mu$, where $\alpha, \beta = +, -$ are the tangent space light-cone indices. The Lagrangian takes the form

$$L_{GS} = Str\left(J^{(2)}_+ J^{(2)}_- + \frac{\kappa}{2}\left(J^{(1)}_+ J^{(3)}_- - J^{(1)}_- J^{(3)}_+\right)\right) e^+ \wedge e^-, \qquad e^+ \wedge e^- = \frac{dx^0 \wedge dx^1}{\det e^\mu_\alpha} \tag{72}$$

and the equations of motion can be written as follows

$$\delta_J L_{GS} = -Str\left(\left(f^{-1}\delta f\right), D_\alpha \Lambda^\alpha\right), \qquad \Lambda^\alpha \equiv \eta^{\alpha\beta} J^{(2)}_\beta - \frac{1}{2}\kappa\epsilon^{\alpha\beta}\left(J^{(1)}_\beta - J^{(3)}_\beta\right), \tag{73}$$

where we have suppressed the term $e^+ \wedge e^-$. The equations of motion for the frame field e imply the Virasoro constrains $Str(J^{(2)}_\pm, J^{(2)}_\pm) = 0$.

To introduce the Lax connection for the sigma model we combine the Maurer-Cartan (MC) and the Euler-Lagrange (EL) equation into a single flat connection. To make an ansatz for the Lax connection it is useful to recast everything in terms of differential forms and see what are the independent current components to be used in the ansatz.

The EL equations of motion (73) can be written, in the form

$$d * J^{(2)} + J^{(0)} \wedge *J^{(2)} + *J^{(2)} \wedge J^{(0)} + \kappa\left(J^{(1)} \wedge J^{(1)} - J^{(3)} \wedge J^{(3)}\right) = 0,$$

$$J^{(1)} \wedge *J^{(2)} + *J^{(2)} \wedge J^{(1)} + \kappa\left(J^{(1)} \wedge J^{(2)} + J^{(2)} \wedge J^{(1)}\right) = 0,$$

$$J^{(3)} \wedge *J^{(2)} + *J^{(2)} \wedge J^{(3)} - \kappa\left(J^{(3)} \wedge J^{(2)} + J^{(2)} \wedge J^{(3)}\right) = 0,$$

where we have used the the MC equations $dJ + J \wedge J = 0$ on the fermionic part and the condition $\kappa^2 = 1$. This is a good representation in the sense that the world-sheet metric h appears only in the term $*J^{(2)}$ through the Hodge star $*$ as in the Lagrangian L_{GS}. We see that the only forms involved are $J^{(0)}, J^{(2)}, *J^{(2)}, J^{(1)}, J^{(3)}$, thus the ansatz is

$$L = l_0 J^{(0)} + l_1 J^{(2)} + l_2 * J^{(2)} + l_3 J^{(1)} + l_4 J^{(3)}, \qquad l_i \in \mathbb{C}. \tag{74}$$

For the curvature we have, after finding $l_0 = 1$, that

$$dL + L \wedge L = c_1 J^{(1)} \wedge J^{(1)} + c_2 J^{(2)} \wedge J^{(2)} + c_3 J^{(3)} \wedge J^{(3)} + c_4\left(J^{(1)} \wedge J^{(2)} + J^{(2)} \wedge J^{(1)}\right) +$$

$$+ c_5\left(J^{(1)} \wedge J^{(3)} + J^{(3)} \wedge J^{(1)}\right) + c_6\left(J^{(2)} \wedge J^{(3)} + J^{(3)} \wedge J^{(2)}\right),$$

where

$$c_1 = l_3^2 - l_1 - \kappa l_2, \qquad c_2 = l_1^2 - l_2^2 - 1, \qquad c_3 = l_4^2 - l_1 + \kappa l_2,$$

$$c_4 = l_1 l_3 - \kappa l_2 l_3 - l_4, \qquad c_5 = l_3 l_4 - 1, \qquad c_6 = l_1 l_4 + \kappa l_2 l_4 - l_3.$$

This connection is flat when

$$l_0 = 1, \qquad l_1 \equiv w, \qquad l_2 = s_2\sqrt{w^2 - 1}, \qquad l_3 = s_3\sqrt{w + \kappa\, l_2}, \qquad l_4 = s_4\sqrt{w - \kappa\, l_2},$$

where s_2, s_3, s_4 can be ± 1 in any combination and w is the only independent complex parameter. The equations for c_4, c_5, c_6 are redundant and trivially satisfied.

In order to make contact with the superalgebra structure of the SSSSG integrable hierarchy, we consider the following complex curve in $\mathbb{C}^2$

$$w^2 \equiv f(z), \qquad f(z) = 1 + \frac{1}{4}\phi(z)^{-1}, \qquad \phi(z)^{-1} = \left(z^2 - z^{-2}\right)^2,$$

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with $\phi(z)$ defined in (66). The relation between the the spectral parameters w, z of the σ-model and the SSSSG model is defined through the curve above inducing the map $l_i(w) \to l_i(z)$, with

$$l_0 \;=\; 1, \qquad l_1 = \; w = s_1 \frac{1}{2}\left(z^2 + z^{-1}\right), \qquad l_2 = s_2 \frac{1}{2}\left(z^2 - z^{-2}\right),$$

$$l_3 \;=\; s_3 \sqrt{\left(\frac{s_1 + \kappa s_2}{2}\right) z^2 + \left(\frac{s_1 - \kappa s_2}{2}\right) z^{-2}}, \qquad l_4 = s_4 \sqrt{\left(\frac{s_1 - \kappa s_2}{2}\right) z^2 + \left(\frac{s_1 + \kappa s_2}{2}\right) z^{-2}},$$

where $s_2 = \pm 1$. To recover the Lax operators (16) consider the particular solution $s_1 = -s_2 = s_3 = s_4 = -\kappa = 1$ implying

$$l_1 - l_2 = z^2, \qquad l_1 + l_2 = z^{-2}, \qquad l_3 = z, \qquad l_4 = z^{-1}$$

and the following form for the σ-model Lax connection

$$L_+ = J_+^{(0)} + z J_+^{(1)} + z^2 J_+^{(2)} + z^{-1} J_+^{(3)}, \qquad L_- = J_-^{(0)} + z^{-1} J_-^{(3)} + z^{-2} J_-^{(2)} + z J_-^{(1)}, \tag{75}$$

which is clearly valued in the twisted affine superalgebra $\widehat{\mathfrak{f}}$ defined in (5). It becomes the SSSSG Lax connection after going to conformal gauge and fixing all the local symmetries by means the Pohlmeyer reduction (recall that $J_+^{(3)} = J_-^{(1)} = 0$)

$$J_+^{(0)} \;=\; \gamma^{-1} \partial_+ \gamma + \gamma^{-1} A_+^{(l)} \gamma, \qquad J_+^{(1)} = \psi_+, \qquad J_+^{(2)} = -\Lambda, \qquad A_+^{(l)} = u^{-1} \partial_+ u, \tag{76}$$

$$J_-^{(0)} \;=\; A_-^{(r)}, \qquad J_-^{(3)} = \gamma^{-1} \psi_- \gamma, \qquad J_-^{(2)} = -\gamma^{-1} \Lambda \gamma, \qquad A_-^{(r)} = \overline{u}^{-1} \partial_- \overline{u}.$$

The key point to show the existence of extended 2d supersymmetry in the phase space of the reduced GS σ-model is to exploit the integrable properties of the SSSSG integrable hierarchy, namely (10) and rewrite the Lax operators in the following two equivalent forms

$$\mathcal{L}_\pm \;=\; \chi \overline{u}^{-1} \left(\partial_\pm - z^{\pm 2} \Lambda\right) \overline{u} \chi^{-1} = \gamma^{-1} \widetilde{\chi} u^{-1} \left(\partial_\pm - z^{\pm 2} \Lambda\right) u \widetilde{\chi}^{-1} \gamma, \tag{77}$$

$$\mathcal{L}_\epsilon \;=\; \chi \overline{u}^{-1} \left(\delta_\epsilon + z\epsilon\right) \overline{u} \chi^{-1} = \gamma^{-1} \widetilde{\chi} u^{-1} \left(\delta_\epsilon + z\epsilon\right) u \widetilde{\chi}^{-1} \gamma,$$

$$\mathcal{L}_{\overline{\epsilon}} \;=\; \chi \overline{u}^{-1} \left(\delta_{\overline{\epsilon}} + z^{-1} \overline{\epsilon}\right) \overline{u} \chi^{-1} = \gamma^{-1} \widetilde{\chi} u^{-1} \left(\delta_{\overline{\epsilon}} + z^{-1} \overline{\epsilon}\right) u \widetilde{\chi}^{-1} \gamma,$$

supplemented by the constraints (17). The operators $\mathcal{L}_\epsilon$ and $\mathcal{L}_{\overline{\epsilon}}$ are responsible for the supersymmetry transformations.

In the next section we construct two possible sets of supersymmetry transformations that, by construction, preserve the equations of motion $F_{+-} = 0$.

Remark 2 *In (75) we have the action of the kappa symmetry transformations and in (77) we have the action of the 2d rigid supersymmetry transformations. The number of supersymmetries and the number of kappa symmetries is the same and perhaps there is a relation between them in the passage from (75) to (77). However, we have not succeeded in showing such a connection.*

5 General supersymmetry variations.

We now proceed to find the supersymmetry variations for all the fields $\gamma, u, \overline{u}, \psi_\pm$. This is an important result and for this reason we do this in some detail.

Consider the Q_H grade ± 1 SUSY flows associated to $b = z\epsilon$, $\epsilon \in \mathfrak{f}_1^\perp$ and $\overline{b} = z^{-1} \overline{\epsilon}$, $\overline{\epsilon} \in \mathfrak{f}_3^\perp$. From the second expression of (77) we have two equivalent representations for $\mathcal{L}_\epsilon$ (recall that $\mathcal{L}_\epsilon$ is valued on Q_H grades 0 and 1 only)

$$\mathcal{L}_\epsilon \;=\; \delta_\epsilon + \overline{u}^{-1} \delta_\epsilon \overline{u} + [y_{-1} + \theta_{-1}, \epsilon_{\overline{u}}] + z\epsilon_{\overline{u}},$$

$$\mathcal{L}_\epsilon \;=\; \delta_\epsilon + \gamma^{-1} \delta_\epsilon \gamma + \gamma^{-1} u^{-1} \delta_\epsilon u \gamma - z\gamma^{-1} \left(\delta_\epsilon y_{+1} + \delta_\epsilon \theta_{+1}\right) \gamma + z\gamma^{-1} \left[y_{+1} + \theta_{+1}, u^{-1} \delta_\epsilon u\right] \gamma + z\gamma^{-1} \epsilon_u \gamma,$$

where we have used the definitions (11) and defined $\epsilon_{\overline{u}} \equiv \overline{u}^{-1}\epsilon\overline{u}, \ \epsilon_u \equiv u^{-1}\epsilon u, \theta_{\pm 1} \in \widehat{\mathfrak{f}}^{\perp}$. By equating we have

$$\begin{aligned}
\gamma^{-1}\delta_\epsilon\gamma &= [y_{-1} + \theta_{-1}, \epsilon_{\overline{u}}] + \overline{u}^{-1}\delta_\epsilon\overline{u} - \gamma^{-1}u^{-1}\delta_\epsilon u\gamma, \\
\delta_\epsilon y_{+1} &= -\left(\gamma\epsilon_{\overline{u}}\gamma^{-1}\right)^{\parallel} + \left[y_{+1}, u^{-1}\delta_\epsilon u\right], \\
\delta_\epsilon\theta_{+1} &= -\left(\gamma\epsilon_{\overline{u}}\gamma^{-1}\right)^{\perp} + \left[\theta_{+1}, u^{-1}\delta_\epsilon u\right] + \epsilon_u,
\end{aligned}$$

where in the second and third lines we have splitted with respect to $\mathfrak{f}^{\parallel}$ and $\mathfrak{f}^{\perp}$. The second line is equivalent to

$$\delta_\epsilon\psi_- = -\left[\Lambda, \gamma\epsilon_{\overline{u}}\gamma^{-1}\right] + \left[\psi_-, u^{-1}\delta_\epsilon u\right].$$

To simplify, we introduce the notation

$$\mathcal{L}_\epsilon = \delta_\epsilon + w + z\epsilon_{\overline{u}}, \qquad w \equiv \overline{u}^{-1}\delta_\epsilon\overline{u} + [y_{-1} + \theta_{-1}, \epsilon_{\overline{u}}]$$

and compute the variations coming from $[\mathcal{L}_\epsilon, \mathcal{L}_+] = 0$. This relation decomposes along Q_H grades 0,1 and 2. The Q_H grade 0 and 2 equation are satisfied while the Q_H grade 1 equation splits along $\mathfrak{f}^{\parallel}$ and $\mathfrak{f}^{\perp}$ with the kernel part also satisfied and we are left with

$$\delta_\epsilon\psi_+ = \left[\left(\gamma^{-1}D_+^{(l)}\gamma\right)^{\parallel}, \epsilon_{\overline{u}}\right] - \left[w^{\perp}, \psi_+\right].$$

Now we compute $[\mathcal{L}_\epsilon, \mathcal{L}_-] = 0$. This splits along Q_H grades -2,-1,0 and 1. The Q_H grade -1,-2 and 1 equations are all satisfied as well as the $\mathfrak{f}^{\parallel}$ part of the Q_H grade 0 equation and we are left with

$$\delta_\epsilon A_-^{(r)} = D_-^{(r)}w^{\perp} + \left[\left(\gamma^{-1}\psi_-\gamma\right)^{\perp}, \epsilon_{\overline{u}}\right].$$

Putting all together we have a raw expression for the SUSY variations

$$\begin{aligned}
\gamma^{-1}\delta_\epsilon\gamma &= -\left[\Lambda, [\psi_+, \epsilon_{\overline{u}}]\right] + w^{\perp} - \gamma^{-1}u^{-1}\delta_\epsilon u\gamma, \\
\delta_\epsilon\psi_+ &= \left[\left(\gamma^{-1}D_+^{(L)}\gamma\right)^{\parallel}, \epsilon_{\overline{u}}\right] - \left[w^{\perp}, \psi_+\right], \\
\delta_\epsilon\psi_- &= -\left[\Lambda, \gamma\epsilon_{\overline{u}}\gamma^{-1}\right] + \left[\psi_-, u^{-1}\delta_\epsilon u\right], \\
\delta_\epsilon A_-^{(R)} &= D_-^{(R)}w^{\perp} + \left[\left(\gamma^{-1}\psi_-\gamma\right)^{\perp}, \epsilon_{\overline{u}}\right],
\end{aligned} \tag{78}$$

where $w^{\perp} = \overline{u}^{-1}\delta_\epsilon\overline{u} + [\theta_{-1}, \epsilon_{\overline{u}}]$. The $\delta_{\overline{\epsilon}}$ SUSY variations are found by parity. We see that there is a large amount of freedom in these expressions and this can be exploited according to our needs. There are two important special cases to be considered below.

5.1 Local supersymmetry variations.

These set of supersymmetry variations was originally introduced by hand in [26] to be symmetries of a variant of the fermionic extension of the perturbed gauge WZNW model associated to the equations of motion (19) in the vector gauge $u = \overline{u}$. Here we deduce this kind of supersymmetry and show that it is a consequence of the integrable hierarchy structure governing the SSSSG models.

The very form of (78) suggest the following choice

$$w^{\perp} = 0, \qquad u^{-1}\delta_\epsilon u = 0$$

and this imply that

$$\overline{u}^{-1}\delta_\epsilon\overline{u} = -[\theta_{-1}, \epsilon_{\overline{u}}], \qquad \delta_\epsilon A_+^{(l)} = 0.$$

The consistency between the last equation of (78) and the $\overline{u}^{-1}\delta_\epsilon\overline{u}$ variation right above (use the identity $\delta_\epsilon A^{(r)}_\pm = D^{(r)}_\pm\left(\overline{u}^{-1}\delta_\epsilon\overline{u}\right)$) allow to determine θ_{-1} for this case and we do not need its explicit form. From the variation $\delta_\epsilon A^{(r)}_-$ we have

$$\delta_\epsilon\overline{u}u^{-1} = \partial_-^{-1}\left\{\overline{u}\left[\left(\gamma^{-1}\psi_-\gamma\right)^\perp,\epsilon_{\overline{u}}\right]\overline{u}^{-1}\right\}.$$

The full set of SUSY variations is then given by

$$\gamma^{-1}\delta_\epsilon\gamma = -\left[\Lambda,\left[\psi_+,\epsilon_{\overline{u}}\right]\right], \qquad \delta_\epsilon\psi_+ = \left[\left(\gamma^{-1}D^{(l)}_+\gamma\right)^\parallel,\epsilon_{\overline{u}}\right], \qquad \delta_\epsilon\psi_- = -\left[\Lambda,\gamma\epsilon_{\overline{u}}\gamma^{-1}\right], \tag{79}$$

$$\delta_\epsilon A^{(l)}_+ = 0, \qquad \delta_\epsilon A^{(r)}_- = \left[\left(\gamma^{-1}\psi_-\gamma\right)^\perp,\epsilon_{\overline{u}}\right],$$

$$u^{-1}\delta_\epsilon u = 0, \qquad \delta_\epsilon\overline{u}u^{-1} = \partial_-^{-1}\left\{\overline{u}\left[\left(\gamma^{-1}\psi_-\gamma\right)^\perp,\epsilon_{\overline{u}}\right]\overline{u}^{-1}\right\},$$

and

$$\delta_{\overline{\epsilon}}\gamma\gamma^{-1} = \left[\Lambda,\left[\psi_-,\overline{\epsilon}_u\right]\right], \qquad \delta_{\overline{\epsilon}}\psi_- = \left[\left(\gamma D^{(r)}_-\gamma^{-1}\right)^\parallel,\overline{\epsilon}_u\right], \qquad \delta_{\overline{\epsilon}}\psi_+ = -\left[\Lambda,\gamma^{-1}\overline{\epsilon}_u\gamma\right], \tag{80}$$

$$\delta_{\overline{\epsilon}}A^{(r)}_- = 0, \qquad \delta_{\overline{\epsilon}}A^{(l)}_+ = \left[\left(\gamma\psi_+\gamma^{-1}\right)^\perp,\overline{\epsilon}_u\right],$$

$$\overline{u}^{-1}\delta_{\overline{\epsilon}}\overline{u} = 0, \qquad \delta_{\overline{\epsilon}}uu^{-1} = \partial_+^{-1}\left\{u\left[\left(\gamma\psi_+\gamma^{-1}\right)^\perp,\overline{\epsilon}_u\right]u^{-1}\right\}.$$

Notice that our construction shows that the natural gauge objects are u and $\overline{u}$ and also explains the integrable origin of these local symmetries.

Now, we want to see if (13) holds. Unfortunately, the price paid for introducing locality is that the SUSY algebra becomes field dependent and highly non-trivial[18]. For the δ_ϵ variations it is given by[19]

$$\left[\delta_\epsilon,\delta_{\epsilon'}\right]\gamma = -2\epsilon\cdot\epsilon'\partial_+\gamma + \delta_{\epsilon_{l/r}}\gamma, \qquad \left[\delta_\epsilon,\delta_{\epsilon'}\right]\psi_+ = -2\epsilon\cdot\epsilon'\partial_+\psi_+ + \delta_{\epsilon_r}\psi_+,$$

$$\left[\delta_\epsilon,\delta_{\epsilon'}\right]\psi_- = -2\epsilon\cdot\epsilon'\partial_+\psi_- + \delta_{\epsilon_l}\psi_-, \qquad \left[\delta_\epsilon,\delta_{\epsilon'}\right]A^{(r)}_- = -2\epsilon\cdot\epsilon'\partial_+A^{(r)}_- + \delta_{\epsilon_r}A^{(r)}_-$$

$$\left[\delta_\epsilon,\delta_{\epsilon'}\right]A^{(l)}_+ = -2\epsilon\cdot\epsilon'\partial_+A^{(l)}_+ + \delta_{\epsilon_l}A^{(l)}_+ = 0, \qquad \epsilon_l \equiv -2\epsilon\cdot\epsilon'A^{(l)}_+, \qquad \epsilon_r \equiv -\left(2\epsilon\cdot\epsilon'A^{(l)}_+ + \widetilde{Q}\right),$$

where we have used the infinitesimal version of (18) given by (take $h_{l/r} = \exp\epsilon_{l/r}$)

$$\delta_{\epsilon_{l/r}}\gamma = \epsilon_l\gamma - \gamma\epsilon_r, \qquad \delta_{\epsilon_r}\psi_+ = \left[\epsilon_r,\psi_+\right], \qquad \delta_{\epsilon_l}\psi_- = \left[\epsilon_l,\psi_-\right],$$

$$\delta_{\epsilon_l}A^{(l)}_+ = -D^{(l)}_+\epsilon_l, \qquad \delta_{\epsilon_r}A^{(r)}_- = -D^{(r)}_-\epsilon_r.$$

For the $\delta_{\overline{\epsilon}}$ variations it is given by[20]

$$\left[\delta_{\overline{\epsilon}},\delta_{\overline{\epsilon}'}\right]\gamma = -2\overline{\epsilon}\cdot\overline{\epsilon}'\partial_-\gamma + \delta_{\epsilon_{l/r}}\gamma, \qquad \left[\delta_{\overline{\epsilon}},\delta_{\overline{\epsilon}'}\right]\psi_- = -2\overline{\epsilon}\cdot\overline{\epsilon}'\partial_-\psi_- + \delta_{\epsilon_l}\psi_-,$$

$$\left[\delta_{\overline{\epsilon}},\delta_{\overline{\epsilon}'}\right]\psi_+ = -2\overline{\epsilon}\cdot\overline{\epsilon}'\partial_-\psi_+ + \delta_{\epsilon_r}\psi_+, \qquad \left[\delta_{\overline{\epsilon}},\delta_{\overline{\epsilon}'}\right]A^{(l)}_+ = -2\overline{\epsilon}\cdot\overline{\epsilon}'\partial_-A^{(l)}_+ + \delta_{\epsilon_l}A^{(l)}_+,$$

$$\left[\delta_{\overline{\epsilon}},\delta_{\overline{\epsilon}'}\right]A^{(r)}_- = -2\overline{\epsilon}\cdot\overline{\epsilon}'\partial_-A^{(r)}_- + \delta_{\epsilon_r}A^{(r)}_- = 0, \qquad \epsilon_r \equiv -2\overline{\epsilon}\cdot\overline{\epsilon}'A^{(r)}_-, \qquad \epsilon_l \equiv -\left(2\overline{\epsilon}\cdot\overline{\epsilon}'A^{(r)}_- + \widetilde{Q}'\right),$$

while for the mixed variations it is

$$\left[\delta_\epsilon,\delta_{\overline{\epsilon}}\right]\gamma = \epsilon_l\gamma - \gamma\epsilon_r, \qquad \left[\delta_\epsilon,\delta_{\overline{\epsilon}}\right]\psi_+ = \delta_{\epsilon_r}\psi_+, \qquad \left[\delta_\epsilon,\delta_{\overline{\epsilon}}\right]\psi_- = \delta_{\epsilon_l}\psi_-,$$

$$\left[\delta_\epsilon,\delta_{\overline{\epsilon}}\right]A^{(l)}_+ = -D^{(l)}_+\epsilon_l, \qquad \left[\delta_\epsilon,\delta_{\overline{\epsilon}}\right]A^{(r)}_- = -D^{(r)}_-\epsilon_r,$$

$$\epsilon_r \equiv -\left[\epsilon_{\overline{u}},\left(\gamma^{-1}\overline{\epsilon}_u\gamma\right)^\perp\right], \qquad \epsilon_l \equiv -\left[\left(\gamma\epsilon_{\overline{u}}\gamma^{-1}\right)^\perp,\overline{\epsilon}_u\right].$$

---

[18] In the rest of the paper we assume that $[\epsilon,\epsilon'] = 2\epsilon\cdot\epsilon'\Lambda$ and $[\overline{\epsilon},\overline{\epsilon}'] = 2\overline{\epsilon}\cdot\overline{\epsilon}'\Lambda$, which is valid for the superalgebras of interest.

[19] The $\mathfrak{f}^\perp$ valued term $\widetilde{Q}$ is given by $\widetilde{Q} = Q - 2\epsilon\cdot\epsilon'\left(A^{(l)}_+ - A^{(r)}_+\right)$, $Q = \mathcal{O} - 2\epsilon\cdot\epsilon'(2\Lambda\psi^2_+)$, $\mathcal{O} = \left[\left[\widetilde{\Lambda}\psi_+,\epsilon_{\overline{u}}\right],\left[\widetilde{\Lambda}\psi_+,\epsilon'_{\overline{u}}\right]\right]$.

[20] The $\mathfrak{f}^\perp$ valued term $\widetilde{Q}'$ is given by $\widetilde{Q}' = Q' - 2\overline{\epsilon}\cdot\overline{\epsilon}'\left(A^{(r)}_- - A^{(l)}_-\right)$, $Q' = \mathcal{O}' - 2\overline{\epsilon}\cdot\overline{\epsilon}'(2\Lambda\psi^2_-)$, $\mathcal{O}' = \left[\left[\widetilde{\Lambda}\psi_-,\overline{\epsilon}_u\right],\left[\widetilde{\Lambda}\psi_-,\overline{\epsilon}'_u\right]\right]$.

Due to the fact that these superalgebra of flows is field dependent and does not satisfy (13), we will not consider it any further, at least in this paper.

5.2 Non-local supersymmetry variations.

These rather simpler transformations were recently introduced in [15] as the on-shell supersymmetry variations associated to the phase space of the Pohlmeyer reduced GSsσ models on $AdS_n \times S^n$, $n = 2, 3, 5$, namely the SSSSG models. See also [16] to see the same results and for further details on soliton solutions and quantization.

These transformations are found in the on-shell gauge $u = \overline{u} = I$. From (78) we have

$$\gamma^{-1}\delta_\epsilon\gamma = -\left[\widetilde{\Lambda}\psi_+, \epsilon\right] + w^\perp, \qquad \delta_\epsilon\psi_+ = \left[\left(\gamma^{-1}\partial_+\gamma\right)^\|, \epsilon\right] - \left[w^\perp, \psi_+\right], \tag{81}$$

$$\delta_\epsilon\psi_- = -\left[\Lambda, \gamma\epsilon\gamma^{-1}\right], \qquad w^\perp = [\theta_{-1}, \epsilon], \qquad \theta_{-1} = -\partial_-^{-1}\left(\gamma^{-1}\psi_-\gamma\right)^\perp,$$

$$\delta_\epsilon\theta_{+1} = -\left(\gamma\epsilon\gamma^{-1}\right)^\perp + \epsilon$$

and

$$\delta_{\overline{\epsilon}}\gamma\gamma^{-1} = \left[\widetilde{\Lambda}\psi_-, \overline{\epsilon}\right] - \overline{w}^\perp, \qquad \delta_{\overline{\epsilon}}\psi_- = -\left[\left(\partial_-\gamma\gamma^{-1}\right)^\|, \overline{\epsilon}\right] - \left[\overline{w}^\perp, \psi_-\right] \tag{82}$$

$$\delta_{\overline{\epsilon}}\psi_+ = -\left[\Lambda, \gamma^{-1}\overline{\epsilon}\gamma\right], \qquad \overline{w}^\perp = [\theta_{+1}, \overline{\epsilon}], \qquad \theta_{+1} = -\partial_+^{-1}\left(\gamma\psi_+\gamma^{-1}\right)^\perp,$$

$$\delta_{\overline{\epsilon}}\theta_{-1} = -\left(\gamma^{-1}\overline{\epsilon}\gamma\right)^\perp + \overline{\epsilon},$$

where θ_{-1}, θ_{+1} were determined by the condition $\delta_\epsilon A_-^{(r)} = 0$, $\delta_{\overline{\epsilon}} A_+^{(l)} = 0$, respectively. These last conditions means that the supersymmetry variations (81),(82) preserve the constraints (44), hence they are symmetries of the SSSSG phase space.

Using these variations we can confirm (13) by explicit action on the fields[21] $\gamma, \psi_\pm$

$$[\delta_\epsilon, \delta_{\epsilon'}](*) = 2\epsilon \cdot \epsilon'\partial_+(*), \qquad [\delta_{\overline{\epsilon}}, \delta_{\overline{\epsilon}'}](*) = 2\overline{\epsilon} \cdot \overline{\epsilon}'\partial_-(*) \tag{83}$$

$$[\delta_\epsilon, \delta_{\overline{\epsilon}}](*) = \delta_h(*), \qquad \delta_h(*) = [h, *], \qquad h \equiv -[\epsilon, \overline{\epsilon}].$$

which is nothing but the algebra of $\mathfrak{s}$ written above in (15). From this we see that the role of the non-local terms $\theta_{\pm 1}$ in the variations is to maintain the isomorphism (13), i.e, $\mathfrak{s} \simeq \widehat{\mathfrak{s}}$. The key step in the proof of the mixed commutator, which is the most involved, are the relations

$$\delta_\epsilon\overline{w}^\perp + \left[\left(\gamma\epsilon\gamma^{-1}\right), \overline{\epsilon}\right] = [\epsilon, \overline{\epsilon}], \qquad \delta_{\overline{\epsilon}}w^\perp - \left[\epsilon, \left(\gamma^{-1}\overline{\epsilon}\gamma\right)^\perp\right] = -[\epsilon, \overline{\epsilon}].$$

6 Moment maps for the action of the superalgebra $\mathfrak{s}$.

In this section we construct the moment maps for the actions on the phase space of the symmetry flows of the sub-superalgebra $\mathfrak{s}$. This leads to another identification of the reduced phase of the SSSSG model and also explains the geometric origin of the non-Abelian R-symmetry group that rotates the DS supercharges.

Consider the symplectic 2-form of the fermionic extension of the perturbed WZNW model. It is given by the inverse of the Poisson bi-vector $\Theta^{(0)}$ defined in (38), i.e, $\Omega_{WZNW} = (\Omega_+ - \Omega_-)$, with

$$-\Omega_+ = \frac{1}{2}\left(\delta\gamma\gamma^{-1} \wedge \partial_+\left(\delta\gamma\gamma^{-1}\right)\right) + \frac{1}{2}\left(\delta\psi \wedge \widetilde{\Lambda}\delta\psi\right), \tag{84}$$

$$-\Omega_- = \frac{1}{2}\left(\gamma^{-1}\delta\gamma \wedge \partial_-\left(\gamma^{-1}\delta\gamma\right)\right) + \frac{1}{2}\left(\delta\overline{\psi} \wedge \widetilde{\Lambda}\delta\overline{\psi}\right),$$

---

[21]This is the full algebra, not the free limit approximation as was considered in [15].

where we have re-scaled with the factor $1/2$. We will need the following identities

$$\delta q = D_+ \left(\gamma^{-1} \delta \gamma \right), \quad \delta \overline{q} = -D_- \left(\delta \gamma \gamma^{-1} \right), \quad \gamma \delta q \gamma^{-1} = \partial_+ \left(\delta \gamma \gamma^{-1} \right), \quad \gamma^{-1} \delta \overline{q} \gamma = -\partial_- \left(\gamma^{-1} \delta \gamma \right),$$

where we have defined $D_-(*) = [\partial_- + \overline{q}, *]$.

This symplectic form is invariant under the action of the chiral $H_l \times H_r$ gauge transformations

$$\widetilde{\gamma} = h_l \gamma h_r, \qquad \widetilde{\psi} = h_r^{-1} \psi h_r, \qquad \widetilde{\overline{\psi}} = h_l \overline{\psi} h_l^{-1} \qquad h_l \equiv h_l(x^-), \qquad h_r \equiv h_r(x^+) \tag{85}$$

and to compute the moments $\mu_{l/r}$ for these actions, we use the following definitions

$$\delta H_\epsilon(*) = -\Omega \left(X_\epsilon, * \right), \qquad H_\epsilon = \int_{-\infty}^{+\infty} dx Str \left(\mu, \epsilon \right), \tag{86}$$

where X_ϵ is the vector field in the phase space induced by the action of the symmetry ϵ, i.e $X_\epsilon = \delta_\epsilon \phi^i$. In our case we have $X_{\epsilon_{l/r}} = \left(\delta_{\epsilon_{l/r}} \gamma, \delta_{\epsilon_{l/r}} \psi, \delta_{\epsilon_{l/r}} \overline{\psi} \right)$ with

$$X_{\epsilon_r} = \left(\gamma \epsilon_r, -\left[\epsilon_r, \psi \right], 0 \right), \qquad X_{\epsilon_l} = \left(\epsilon_l \gamma, 0, \left[\epsilon_l, \overline{\psi} \right] \right),$$

where we have taken $h_{l/r} = \exp \epsilon_{l/r}$ in (85). Thus, we need to compute the contractions

$$-\Omega_{WZNW}(X_{\epsilon_r}, *) = -\Omega_+(X_{\epsilon_r}, *), \qquad -\Omega_{WZNW}(X_{\epsilon_l}, *) = \Omega_-(X_{\epsilon_l}, *).$$

Inserting X_{ϵ_r} in the first equation of (84) yields[22]

$$-\Omega_+(X_{\epsilon_r}, *) = \left(\gamma \epsilon_r \gamma^{-1}, \partial_+ \left(\delta \gamma \gamma^{-1} \right) \right) - \left(\left[\epsilon_r, \psi \right], \widetilde{\Lambda} \delta \psi \right).$$

Now, using $\delta q = \gamma^{-1} \partial_+ \left(\delta \gamma \gamma^{-1} \right) \gamma$ and $\delta(2 \Lambda \psi \psi) = -\left[\psi, 2 \Lambda \delta \psi \right]$ we have

$$-\Omega_+(X_{\epsilon_r}, *) = \delta \int_{-\infty}^{+\infty} dx Str \left(q + 2 \Lambda \psi^2, \epsilon_r \right)$$

and from this we identify, modulo δ-exact terms, the moment for the chiral right gauge action

$$\mu_r = q^\perp + 2 \Lambda \psi^2.$$

In a similar way we obtain

$$\Omega_-(X_{\epsilon_l}, *) = \delta \int_{-\infty}^{+\infty} dx Str \left(\overline{q} + 2 \Lambda \overline{\psi}^2, -\epsilon_l \right) \to \mu_l = \left(\overline{q}^\perp + 2 \Lambda \overline{\psi \psi} \right).$$

The moments $\mu_{l/r}$ are precisely the constraints (17) naturally imposed by the dressing formalism of the SSSSG model (10). From this we see that the SSSSG model corresponds to the Hamiltonian reduction of the perturbed WZNW model by the chiral symmetry (85), i.e, $\mu_{l/r} = 0$, see [27]. Hence the importance of restricting the Poisson structures to these level sets. They also affect the form of the DS charges as shown in (27). By contracting again we easily show that

$$\Omega_{WZNW}(X_{\epsilon_l}, X_{\epsilon_l'}) = \int_{-\infty}^{+\infty} dx Str \left(\mu_l, [\epsilon_l, \epsilon_l'] \right), \quad \Omega_{WZNW}(X_{\epsilon_r}, X_{\epsilon_r'}) = \int_{-\infty}^{+\infty} dx Str \left(\mu_r, [\epsilon_r, \epsilon_r'] \right)$$

---

[22]Recall that for left-right invariant vector fields $X_{l/r}$ we have $\delta \gamma(X_l) = \gamma X_l(e)$, $\delta \gamma(X_r) = X_r(e) \gamma$. Identifying $X_l = \gamma \epsilon_r$, $X_r = \epsilon_l \gamma$ and $X_l(e) = \epsilon_r$, $X_r(e) = \epsilon_l$ we obtain $\delta \gamma(X) = \epsilon_l \gamma + \gamma \epsilon_r$.

with the mixed contraction vanishing.

Now, we proceed to compute the moments for the supersymmetries (81), (82) which is the novel part here. We assume that the invariance of the symplectic form holds as a consequence of the invariance of the Lagrangian [15]. The advantage of this computation is that it shows that the non-local SUSY variations are Hamiltonian flows because they obey (86).

Let us write again the δ_ϵ field variations (81) in a more compact form

$$
\begin{aligned}
\delta_\epsilon \gamma &= \gamma \widehat{\epsilon}, \qquad \widehat{\epsilon} \equiv -\left[\widetilde{\Lambda}\psi, \epsilon\right] + w^\perp, \\
\delta_\epsilon \psi &= \left[q^\parallel, \epsilon\right] - \left[w^\perp, \psi\right], \qquad \delta_\epsilon \overline{\psi} = -\left[\Lambda, \gamma\epsilon\gamma^{-1}\right].
\end{aligned}
$$

Using the ψ fermion equation of motion (19) and the explicit form of $w^\perp$ we can show that $\partial_- \widehat{\epsilon} = \left[\epsilon, \gamma^{-1}\overline{\psi}\gamma\right]$, which will be used below. What we need to compute is the following contraction

$$
-\Omega_{WZNW}(X_\epsilon, *) = -\Omega_+(X_\epsilon, *) + \Omega_-(X_\epsilon, *). \tag{87}
$$

Denoting by $X_\epsilon = \left(\delta_\epsilon\gamma, \delta_\epsilon\psi, \delta_\epsilon\overline{\psi}\right)$ and using $\delta\gamma(X_\epsilon) = \gamma\widehat{\epsilon}$, $\delta\psi(X_\epsilon) = \left[q^\parallel, \epsilon\right] - \left[w^\perp, \psi\right]$, $\delta\overline{\psi}(X_\epsilon) = -\left[\Lambda, \gamma\epsilon\gamma^{-1}\right]$ we get

$$
\begin{aligned}
-\Omega_+(X_\epsilon, *) &= \left(\gamma\,\widehat{\epsilon}\gamma^{-1}, \partial_+\left(\delta\gamma\gamma^{-1}\right)\right) + \left(\left[q^\parallel, \epsilon\right] - \left[w^\perp, \psi\right], \widetilde{\Lambda}\delta\psi\right) \tag{88}\\
&= \left(\widehat{\epsilon}, \delta q\right) - \left(\epsilon, \left[q, \widetilde{\Lambda}\delta\psi\right]\right) - \left(w^\perp, \left[\psi, \widetilde{\Lambda}\delta\psi\right]\right) \\
&= \delta\left(\epsilon, \left[\widetilde{\Lambda}\psi, q\right]\right) + \left(w^\perp, \delta\left(q + 2\Lambda\psi\psi\right)\right) \\
&= \delta\left(\epsilon, \left[\widetilde{\Lambda}\psi, q\right]\right) + \left(\delta\mu_R, w^\perp\right).
\end{aligned}
$$

In a similar way, we have

$$
\begin{aligned}
-\Omega_-(X_\epsilon, *) &= \left(\widehat{\epsilon}, \partial_-\left(\gamma^{-1}\delta\gamma\right)\right) - \left(\left[\Lambda, \gamma\epsilon\gamma^{-1}\right], \widetilde{\Lambda}\delta\overline{\psi}\right) \tag{89}\\
&= -\left(\partial_-\widehat{\epsilon}, \gamma^{-1}\delta\gamma\right) - \left(\left[\Lambda, \gamma\epsilon\gamma^{-1}\right], \widetilde{\Lambda}\delta\overline{\psi}\right) \\
&= -\left(\epsilon, \gamma^{-1}\delta\overline{\psi}\gamma + \left[\gamma^{-1}\overline{\psi}\gamma, \gamma^{-1}\delta\gamma\right]\right) \\
&= -\delta\left(\epsilon, \gamma^{-1}\overline{\psi}\gamma\right).
\end{aligned}
$$

Now, at fixed time $(dx^\pm = \pm dx)$ we find that

$$
-\Omega_+(X_\epsilon, *) = \delta\int_{-\infty}^{+\infty} dx\,Str\left(\epsilon, \left[\widetilde{\Lambda}\psi, q\right]\right) + \int_{-\infty}^{+\infty} dx\,Str\left(\delta\mu_R, w^\perp\right), \qquad \Omega_-(X_\epsilon, *) = -\delta\int_{-\infty}^{+\infty} dx\,Str\left(\epsilon, \gamma^{-1}\overline{\psi}\gamma\right).
$$

Using (87) and repeating the computations for (82) we obtain, modulo δ-exact terms, the SUSY moments, i.e, the supercurrent densities of (27), for the δ_ϵ and $\delta_{\overline{\epsilon}}$ susy actions

$$
\mu = \left[\widetilde{\Lambda}q, \psi\right] - \left(\gamma^{-1}\overline{\psi}\gamma\right)^\perp, \qquad \overline{\mu} = \left[\widetilde{\Lambda}\overline{q}, \overline{\psi}\right] - \left(\gamma\psi\gamma^{-1}\right)^\perp. \tag{90}
$$

Note that we can define moments for this rigid supersymmetry only when $\mu_{l/r} = 0$, which is precisely when we restrict ourselves to the reduced phase space in consistency with the Drinfeld-Sokolov procedure. Note also that the non local terms in the SUSY variations are crucial for the result. As above, we perform a second contraction to find

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the action of the gauge group on μ and $\overline{\mu}$

$$\Omega_{WZNW}(X_{\epsilon_r}, X_\epsilon) = \int_{-\infty}^{+\infty} dx Str\left(\mu, [\epsilon_r, \epsilon]\right) + \int_{-\infty}^{+\infty} dx Str\left(\mu_r, [\epsilon_r, w^\perp]\right), \qquad (91)$$

$$\Omega_{WZNW}(X_{\epsilon_l}, X_{\overline{\epsilon}}) = \int_{-\infty}^{+\infty} dx Str\left(\overline{\mu}, [-\epsilon_l, \overline{\epsilon}]\right) + \int_{-\infty}^{+\infty} dx Str\left(\mu_l, [-\epsilon_l, \overline{w}^\perp]\right)$$

with the other mixed brackets vanishing.

Recall that besides (85), we have the action of the residual group H of global gauge transformations

$$\widetilde{\gamma} = u\gamma v^{-1}, \qquad \widetilde{\psi} = v\psi v^{-1}, \qquad \widetilde{\overline{\psi}} = u\overline{\psi}u^{-1}$$

and under this action we can check explicitly that the moments obey the usual equivariance property $\mu(g \cdot \phi) = (Ad^*g)\,\mu(\phi) = g\mu(\phi)g^{-1}$. Taking $g \cdot \phi = \widetilde{\phi}$ and $\phi^i = (\gamma, \psi, \overline{\psi})$ we have

$$\mu_r(\widetilde{\phi}) = (Ad^*v)\,\mu_r(\phi), \qquad \mu(\widetilde{\phi}) = (Ad^*v)\,\mu(\phi),$$
$$\mu_l(\widetilde{\phi}) = (Ad^*u)\,\mu_l(\phi), \qquad \overline{\mu}(\widetilde{\phi}) = (Ad^*u)\,\overline{\mu}(\phi),$$

cf (91) with $\epsilon_r = \epsilon_r = cte$. The supercharges transform under the global gauge group H or R-symmetry group. Note that the residual gauge transformations become the stabilizer of the moments $\mu_{l/r}$ only when they vanish $\mu_{l/r} = 0$, becoming a symmetry of the symplectic quotient $\mathcal{P}$, defined right below, and with coordinates (20). The non-Abelian R-symmetry action is just the equivariance of the susy moments and the SSSSG symplectic form on the reduced phase space $\mathcal{P}$ are now

$$\Omega_{SSSSG} = \Omega_{WZNW}\mid_{\mu_{l/r}=0}, \qquad \mathcal{P} = \widehat{\mu}^{-1}(0)//H, \qquad (92)$$

where $\widehat{\mu} \equiv (\mu_l, \mu_r)$. Clearly, this way of writing $\mathcal{P}$ is equivalent to the co-adjoint orbit formulation of section 2.

The moments for the light-cone translations $x^\pm \to x^\pm + \epsilon^\pm$ can be extracted directly from (27), by writing $Str\left(\Lambda, q_{\pm 2}\right) = \int_{-\infty}^{+\infty} dx Str\left(\mu_\pm, \Lambda^\pm\right), \Lambda^\pm \equiv \epsilon^\pm \Lambda$. Their explicit form can be easily found from the $T_{\mu\nu}$ components written in below in section 7.1.

To finish, we write $\mathfrak{s}$ and its associated moment maps $\mu : \mathcal{P} \to \mathfrak{s}^*$:

$$\mathfrak{s} = \left(\mathfrak{h} \oplus \mathfrak{f}_1^\perp \oplus \mathfrak{f}_3^\perp\right) \ltimes \mathbb{R}^2, \qquad \mu_{l/r} \in \mathfrak{h}^*, \qquad \mu \in (\mathfrak{f}_1^\perp)^*, \qquad \overline{\mu} \in (\mathfrak{f}_3^\perp)^*, \qquad \mu_\pm \in \left(\mathbb{R}^2\right)^*. \qquad (93)$$

7 The 2d supersymmetry algebra with kink central charges.

We will find, under the second Poisson structure (39), the full[23] Poisson superalgebra satisfied by the supercharges $q(b)$ and $\overline{q}(\overline{b})$ with $b = z\epsilon, z^2\Lambda$ and $\overline{b} = z^{-1}\overline{\epsilon}, z^{-2}\Lambda$. There are no charges associated to $k \in \mathfrak{h}$ because of (27) and (44), hence all the brackets are computed on (92) with $k \in \mathfrak{h}$ generating the R-symmetry group. The generators $(b, \overline{b}, k)$ span the sub-superalgebra $\widehat{\mathfrak{s}}$ of (14) and the Poisson superalgebra is that of the symmetries of $\mathfrak{s}$ of (15). As one would expect from (83) and (21), the boundary central charge Z of the mixed bracket $\{q(z\epsilon), \overline{q}(z^{-1}\overline{\epsilon})\}_2 \sim Z$ should be related to the kink electric charge q_0 of the global gauge symmetry H but, as we shall see, Z and q_0 are not related at all and in turn, Z turns out to be a generalization of the well known surface terms of the $N = 1$ and $N = 2$ sine-Gordon and complex sine-Gordon model superalgebras [28] and [29], respectively.

---

[23]In [15] only half of the SUSY algebra were found, here we compute the whole algebra.

7.1 The superalgebra using the Poisson bi-vector Θ_{SSSSG}.

From (27), define $q(z\epsilon) \equiv Str(\epsilon, q_{-1})$, $\overline{q}(z^{-1}\overline{\epsilon}) \equiv Str(\overline{\epsilon}, q_{+1})$ and recall $q(z^2\Lambda) = Str(\Lambda, q_{-2})$ and $\overline{q}(z^{-2}\Lambda) = Str(\Lambda, q_{-2})$. Using (40) we have

$$q(z\epsilon) = \int_{-\infty}^{+\infty} dx\, (j_+(z\epsilon) + j_-(z\epsilon)), \qquad q(z^2\Lambda) = \int_{-\infty}^{+\infty} dx\, (j_+(z^2\Lambda) + j_-(z^2\Lambda)),$$

$$\overline{q}(z^{-1}\overline{\epsilon}) = \int_{-\infty}^{+\infty} dx\, (\overline{j}_+(z^{-1}\overline{\epsilon}) + j_-(z^{-1}\overline{\epsilon})), \qquad \overline{q}(z^{-2}\Lambda) = \int_{-\infty}^{+\infty} dx\, (\overline{j}_+(z^{-2}\Lambda) + \overline{j}_-(z^{-2}\Lambda)),$$

where $T_{++} \equiv j_+(z^2\Lambda)$, $T_{-+} \equiv j_-(z^2\Lambda)$, $T_{+-} \equiv \overline{j}_+(z^{-2}\Lambda)$ and $T_{--} \equiv \overline{j}_-(z^{-2}\Lambda)$. We use the brackets (42) and (43) in the computation.

One important comment is in order here. It is not difficult to show that the brackets $\{q(z\epsilon), q(z\epsilon')\}_2$ and $\{\overline{q}(z^{-1}\overline{\epsilon}), \overline{q}(z^{-1}\overline{\epsilon}')\}_2$ are insensitive to the action of (25), hence they take the same value along the orbits (26) and the computation can be done by using the local diffentials (41). The subtlety appears with the mixed bracket $\{q(z\epsilon), \overline{q}(z^{-1}\overline{\epsilon})\}_2$, which is not invariant. In this case we perform a compensating gauge transformation $\eta_{\pm}$ with $\beta_{\pm 1} = \theta_{\pm 1}$, where $\theta_{\pm 1}$ are defined by (81),(82).

Let us consider first the bracket of $q(z\epsilon)$ with itself. This bracket was already computed in [15], then we pose only the items necessary for compute it. From (41) we find the supercurrent differentials

$$d_q\, j_+(z\epsilon) = [y_{-1}, \epsilon], \qquad d_\psi j_+(z\epsilon) = [y_{-2}, \epsilon] + \frac{1}{2}[y_{-1}, [y_{-1}, \epsilon]],$$

$$d_{\overline{q}}\, j_-(z\epsilon) = -[y_{-1}, \epsilon], \qquad d_{\overline{\psi}}j_-(z\epsilon) = -\epsilon.$$

Using them, we have

$$\{(j_+(z\epsilon)), (j_+(z\epsilon'))\}_2(\mathcal{L}_+) = -(D_+, [d_q\, j_+(z\epsilon), d_q\, j_+(z\epsilon')]) - (\Lambda, [d_\psi j_+(z\epsilon), d_\psi j_+(z\epsilon')]),$$

$$(D_+, [d_q\, j_+(z\epsilon), d_q\, j_+(z\epsilon')]) = -2\epsilon \cdot \epsilon' \left(\Lambda\psi\partial_+\psi + (q^\perp)^2\right),$$

$$(\Lambda, [d_\psi j_+(z\epsilon), d_\psi j_+(z\epsilon')]) = -2\epsilon \cdot \epsilon' \left(\frac{1}{2}(q^\|)^2\right),$$

where we have used $[\epsilon, \epsilon'] = 2\epsilon \cdot \epsilon'\Lambda$. Then, we get

$$\{(j_+(z\epsilon)), (j_+(z\epsilon'))\}_2(\mathcal{L}_+) = -2\epsilon \cdot \epsilon' \int_{-\infty}^{+\infty} dx^+ T_{++}, \qquad T_{++} = -Str\left(\frac{1}{2}(q^\|)^2 + (q^\perp)^2 + \Lambda\psi\partial_+\psi\right).$$

In a similar way we have

$$\{(j_-(z\epsilon)), (j_-(z\epsilon'))\}_2(\mathcal{L}_-) = -(\partial_-, [d_{\overline{q}}\, j_-(z\epsilon), d_{\overline{q}}\, j_-(z\epsilon')]) - \left(\gamma^{-1}\Lambda\gamma, \left[d_{\overline{\psi}}j_-(z\epsilon), d_{\overline{\psi}}j_-(z\epsilon')\right]\right),$$

$$(\partial_-, [d_{\overline{q}}\, j_-(z\epsilon), d_{\overline{q}}\, j_-(z\epsilon')]) = 2\epsilon \cdot \epsilon' \left(\frac{1}{2}\gamma^{-1}\overline{\psi}\gamma\psi\right),$$

$$\left(\gamma^{-1}\Lambda\gamma, \left[d_{\overline{\psi}}j_-(z\epsilon), d_{\overline{\psi}}j_-(z\epsilon')\right]\right) = 2\epsilon \cdot \epsilon' \left(\gamma^{-1}\Lambda\gamma\Lambda\right)$$

and

$$\{(j_-(z\epsilon)), (j_-(z\epsilon'))\}_2(\mathcal{L}_-) = -2\epsilon \cdot \epsilon' \int_{-\infty}^{+\infty} dx^- T_{-+}, \qquad T_{-+} = Str\left(\gamma^{-1}\Lambda\gamma\Lambda + \frac{1}{2}\gamma^{-1}\overline{\psi}\gamma\psi\right).$$

At fixed time $(dx^\pm \to \pm dx)$ we have

$$\{q(z\epsilon), q(z\epsilon')\}_2(\mathcal{L}_x) = \{(j_+(z\epsilon)), (j_+(z\epsilon'))\}_2(\mathcal{L}_+) - \{(j_-(z\epsilon)), (j_-(z\epsilon'))\}_2(\mathcal{L}_-)$$

$$= -2\epsilon \cdot \epsilon' \int_{-\infty}^{+\infty} dx\, (T_{++} + T_{-+}) = -q\left(z^2[\epsilon, \epsilon']\right),$$

and we have shown that

$$\{q(z\epsilon), q(z\epsilon')\}_2 (\mathcal{P}) = -q\left(z^2 [\epsilon, \epsilon']\right). \tag{94}$$

By parity we have

$$\left\{\overline{q}(z^{-1}\overline{\epsilon}), \overline{q}(z^{-1}\overline{\epsilon}')\right\}_2 (\mathcal{L}') = -2\overline{\epsilon} \cdot \overline{\epsilon}' \int_{-\infty}^{+\infty} dx \, (T_{+-} + T_{--}) = -\overline{q}\left(z^{-2} [\overline{\epsilon}, \overline{\epsilon}']\right)$$

and we have that

$$\left\{\overline{q}(z^{-1}\overline{\epsilon}), \overline{q}(z^{-1}\overline{\epsilon}')\right\}_2 (\mathcal{P}) = -\overline{q}\left(z^{-2} [\overline{\epsilon}, \overline{\epsilon}']\right). \tag{95}$$

Now we compute the mixed bracket of $q(z\epsilon)$ with $\overline{q}(z^{-1}\overline{\epsilon})$, which is the novel part here. We need to find

$$
\begin{aligned}
\left\{(j_+(z\epsilon)), (\overline{j}_+(z^{-1}\overline{\epsilon}))\right\}_2 (\mathcal{L}_+) &= -\left(D_+, [d_q \, j_+(z\epsilon), d_q \, \overline{j}_+(z^{-1}\overline{\epsilon})]\right) - \left(\Lambda, [d_\psi j_+(z\epsilon), d_\psi \overline{j}_+(z^{-1}\overline{\epsilon})]\right), \\
\left\{(j_-(z\epsilon)), (\overline{j}_-(z^{-1}\overline{\epsilon}))\right\}_2 (\mathcal{L}_-) &= -\left(\partial_-, [d_{\overline{q}} \, j_-(z\epsilon), d_{\overline{q}} \, \overline{j}_-(z^{-1}\overline{\epsilon})]\right) - \left(\gamma^{-1}\Lambda\gamma, \left[d_{\overline{\psi}}j_-(z\epsilon), d_{\overline{\psi}}\overline{j}_-(z^{-1}\overline{\epsilon})\right]\right).
\end{aligned}
$$

The differentials (41) including the action of (25) are given by

$$
\begin{aligned}
d_\psi j_+(z\epsilon) &= [y_{-2} + \theta_{-2}, \epsilon] + \frac{1}{2}[y_{-1}, [y_{-1}, \epsilon]] + \frac{1}{2}[\theta_{-1}, [\theta_{-1}, \epsilon]] + [y_{-1}, [\theta_{-1}, \epsilon]], \\
d_q \, \overline{j}_+(z^{-1}\overline{\epsilon}) &= -\gamma^{-1}[y_{+1} + \theta_{+1}, \overline{\epsilon}]\gamma, \qquad d_q \, j_+(z\epsilon) = [y_{-1} + \theta_{-1}, \epsilon], \qquad d_\psi \overline{j}_+(z^{-1}\overline{\epsilon}) = -\gamma^{-1}\overline{\epsilon}\gamma
\end{aligned}
$$

and

$$
\begin{aligned}
d_{\overline{q}} \, j_-(z\epsilon) &= -[y_{-1} + \theta_{-1}, \epsilon], \qquad d_{\overline{q}} \, \overline{j}_-(z^{-1}\overline{\epsilon}) = \gamma^{-1}[y_{+1} + \theta_{+1}, \overline{\epsilon}]\gamma, \qquad d_{\overline{\psi}}j_-(z\epsilon) = -\epsilon, \\
d_{\overline{\psi}}\overline{j}_-(z^{-1}\overline{\epsilon}) &= \gamma^{-1}\left([y_{+2} + \theta_{+2}, \overline{\epsilon}] + \frac{1}{2}[y_{+1}, [y_{+1}, \overline{\epsilon}]] + \frac{1}{2}[\theta_{+1}, [\theta_{+1}, \overline{\epsilon}]] + [y_{+1}, [\theta_{+1}, \overline{\epsilon}]]\right)\gamma.
\end{aligned}
$$

Let us compute the easiest term first

$$\left(\Lambda, [d_\psi j_+(z\epsilon), d_\psi \overline{j}_+(z^{-1}\overline{\epsilon})]\right) = -\left(q^\parallel, [\epsilon, \gamma^{-1}\overline{\epsilon}\gamma]\right) - \left(\psi, [[\theta_{-1}, \epsilon], \gamma^{-1}\overline{\epsilon}\gamma]\right).$$

Now we consider

$$
\begin{aligned}
\left(D_+, [d_q \, j_+(z\epsilon), d_q \, \overline{j}_+(z^{-1}\overline{\epsilon})]\right) &= -\left([D_+, d_q \, \overline{j}_+(z^{-1}\overline{\epsilon})], d_q \, j_+(z\epsilon)\right) \\
&= \left(\psi, [[y_{-1} + \theta_{-1}, \epsilon], \gamma^{-1}\overline{\epsilon}\gamma]\right),
\end{aligned}
$$

where we have used

$$\left[D_+, d_q \, \overline{j}_+(z^{-1}\overline{\epsilon})\right] = [\psi, \gamma^{-1}\overline{\epsilon}\gamma], \qquad \partial_+ (y_{+1} + \theta_{+1}) = -\gamma\psi\gamma^{-1}.$$

Then, we have

$$\left\{(j_+(z\epsilon)), (\overline{j}_+(z^{-1}\overline{\epsilon}))\right\}_2 (\mathcal{L}_+) = \int_{-\infty}^{+\infty} dx^+ \, Str\left(q, [\epsilon, \gamma^{-1}\overline{\epsilon}\gamma]\right).$$

On the other hand we have

$$
\begin{aligned}
\left(\gamma^{-1}\Lambda\gamma, \left[d_{\overline{\psi}}j_-(z\epsilon), d_{\overline{\psi}}\overline{j}_-(z^{-1}\overline{\epsilon})\right]\right) &= -\left(\overline{q}^\parallel, [\gamma\epsilon\gamma^{-1}, \overline{\epsilon}]\right) - \left(\overline{\psi}, [\gamma\epsilon\gamma^{-1}, [\theta_{+1}, \overline{\epsilon}]]\right), \\
\left(\partial_-, [d_{\overline{q}} \, j_-(z\epsilon), d_{\overline{q}} \, \overline{j}_-(z^{-1}\overline{\epsilon})]\right) &= \left(\overline{\psi}, [\gamma\epsilon\gamma^{-1}, [y_{+1} + \theta_{+1}, \overline{\epsilon}]]\right),
\end{aligned}
$$

where we have used

$$[\partial_-, d_{\overline{q}} \, j_-(z\epsilon)] = [\gamma^{-1}\overline{\psi}\gamma, \epsilon], \qquad \partial_- (y_{-1} + \theta_{-1}) = -\gamma^{-1}\overline{\psi}\gamma.$$

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Thus, we get

$$\left\{\left(j_-(z\epsilon)\right),\left(\overline{j}_-(z^{-1}\overline{\epsilon})\right)\right\}_2(\mathcal{L}_-) = \int_{-\infty}^{+\infty} dx^- Str\left(\overline{q},\left[\gamma\epsilon\gamma^{-1},\overline{\epsilon}\right]\right).$$

Putting all together we have, at fixed time $(dx^\pm \to \pm dx)$, that

$$
\begin{aligned}
\left\{q(z\epsilon),\overline{q}(z^{-1}\overline{\epsilon})\right\}_2(\mathcal{L}_x) &= \left\{\left(j_+(z\epsilon)\right),\left(\overline{j}_+(z^{-1}\overline{\epsilon})\right)\right\}_2(\mathcal{L}_+) - \left\{\left(j_-(z\epsilon)\right),\left(\overline{j}_-(z^{-1}\overline{\epsilon})\right)\right\}_2(\mathcal{L}_-) \\
&= \int_{-\infty}^{+\infty} dx\,\partial_x Str\left(\gamma\epsilon\gamma^{-1}\overline{\epsilon}\right) = Z_{\epsilon,\overline{\epsilon}}, \qquad Z_{\epsilon,\overline{\epsilon}} \equiv Str\left(\gamma\epsilon\gamma^{-1}\overline{\epsilon}\right)|_{-\infty}^{+\infty}
\end{aligned}
$$

and we have shown that[24]

$$\left\{q(z\epsilon),\overline{q}(z^{-1}\overline{\epsilon})\right\}_2(\mathcal{P}) = Z_{\epsilon,\overline{\epsilon}}. \tag{96}$$

Let us consider $Z_{\epsilon,\overline{\epsilon}}$ for some particular models in order to have a better feeling about its meaning[25]. First we take[26] $\epsilon = \epsilon_i f_i^{(1)}, \overline{\epsilon} = \overline{\epsilon}_j f_j^{(3)}$ and rewrite

$$Str\left(\gamma\epsilon\gamma^{-1}\overline{\epsilon}\right) = \epsilon_i\overline{\epsilon}_j K_{ij}, \qquad K_{ij} = Str\left(\gamma f_i^{(1)}\gamma^{-1}, f_j^{(3)}\right).$$

For the Pohlmeyer reduced $AdS_2 \times S^2$ σ-model we have

$$K = 2\begin{pmatrix} \cos\varphi\cosh\phi & -\sin\varphi\sinh\phi \\ \sin\varphi\sinh\phi & \cos\varphi\cosh\phi \end{pmatrix}.$$

The $N = (1,1)$ sine-Gordon model is obtained by taking $\phi = 0$ and by supressing some fermions and SUSY parameters. Putting $\phi = 0$ above, we have $K_{SG} = 2\mu I \cos\varphi$, where we have re-installed the mass parameter μ. In this case the bracket (96) reduces to the well known result [28] (if we take $\epsilon_1 = \overline{\epsilon}_1 = 0$ or $\epsilon_2 = \overline{\epsilon}_2 = 0$)

$$\left\{q(z\epsilon),\overline{q}(z^{-1}\overline{\epsilon})\right\}_2(\mathcal{P}) = \epsilon\cdot\overline{\epsilon}\left(2\mu\cos\varphi\,|_{-\infty}^{+\infty}\right).$$

For the Pohlmeyer reduced $AdS_3 \times S^3$ models we find

$$K = 2\begin{pmatrix} \cos\varphi\cosh\phi\sin(\theta-\chi) & \cos\varphi\cosh\phi\cos(\theta-\chi) & -\sin\varphi\sinh\phi\cos(t-t') & \sin\varphi\sinh\phi\sin(t-t') \\ -\cos\varphi\cosh\phi\cos(\theta-\chi) & \cos\varphi\cosh\phi\sin(\theta-\chi) & -\sin\varphi\sinh\phi\sin(t-t') & -\sin\varphi\sinh\phi\cos(t-t') \\ \sin\varphi\sinh\phi\cos(t-t') & \sin\varphi\sinh\phi\sin(t-t') & \cos\varphi\cosh\phi\sin(\theta-\chi) & -\cos\varphi\cosh\phi\cos(\theta-\chi) \\ -\sin\varphi\sinh\phi\sin(t-t') & \sin\varphi\sinh\phi\cos(t-t') & \cos\varphi\cosh\phi\cos(\theta-\chi) & \cos\varphi\cosh\phi\sin(\theta-\chi) \end{pmatrix}.$$

At spatial infinity $x \to \pm\infty$ we have that $\gamma \to \gamma_0 \in H$ which is satisfied by the vacuum values $\phi(\pm\infty) = 0$ and $\varphi(\pm\infty) = \pi n_\pm$, $n_\pm \in \mathbb{Z}$. Then, we can set $K_{AdS_3\times S^3} = diag(K_+, K_-)$, where[27]

$$K_\pm = 2\mu\cos\varphi\left(I\sin(\theta-\chi) \pm i\sigma^2\cos(\theta-\chi)\right).$$

Strictly speaking, the $N = (2,2)$ complex sine-Gordon model cannot be obtained by truncation [2], but for illustrative purposes we ignore this and take the limit[28] $\phi = \chi = t = 0$. In this case we have that $K_{cSG} = diag(\overline{K}_+, \overline{K}_-)$, where

---

[24]If we compute $\left\{q(z\epsilon),\overline{q}(z^{-1}\overline{\epsilon})\right\}_2(\mathcal{L}'_x)$, we get the same answer.

[25]Use the results of [15] with $B \to \gamma$, $F_{1i} \to f_i^{(1)}$ and $F_{3j} \to f_j^{(3)}$.

[26]The supraindexes $(1), (3)$ denote the $\mathbb{Z}_4$ grading.

[27]The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[28]This limit corresponds to $\gamma = diag(\gamma_A, \gamma_S) \to diag(I_2, \gamma_S)$, where A, S refers to the AdS and S spaces.

$\overline{K}_\pm = K_\pm(\chi = 0)$. If we take $\epsilon_1 = \epsilon_2 = \overline{\epsilon}_1 = \overline{\epsilon}_2 = 0$ or $\epsilon_3 = \epsilon_4 = \overline{\epsilon}_3 = \overline{\epsilon}_4 = 0$ we use $\overline{K}_-$ or $\overline{K}_+$ respectively. For example, if we choose $\overline{K}_+$ and rotate it to $K' = M^\dagger \overline{K}_+ M$ we recover the well known result [29]

$$\left\{ q(z\epsilon)', \overline{q}(z^{-1}\overline{\epsilon})' \right\}_2 = \epsilon_i \overline{\epsilon}_j K'_{ij} \mid^{+\infty}_{-\infty}, \qquad K' = -2\mu i \cos\varphi \left(iI \sin\theta + \sigma^3 \cos\theta \right),$$

where $M = - \left(\sigma^2 - \sigma^3 \right) / \sqrt{2}$. Thus, we conclude that $Z_{\epsilon,\overline{\epsilon}}$ is the general expression for the central charge of the 2d supersymmetry algebra associated to the Pohlmeyer reduced GS σ-models on $AdS_n \times S^2$, $n = 2, 3, 5$.

Finally, we write the Poisson superalgebra associated to the action $\mathfrak{s} \circlearrowleft \mathcal{P}$

$$\begin{aligned}
\left\{ q(z\epsilon), q(z\epsilon') \right\}_2 (\mathcal{P}) &= -q \left(z^2 \left[\epsilon, \epsilon' \right] \right), \qquad \left\{ \overline{q}(z^{-1}\overline{\epsilon}), \overline{q}(z^{-1}\overline{\epsilon}') \right\}_2 (\mathcal{P}) = -\overline{q} \left(z^{-2} \left[\overline{\epsilon}, \overline{\epsilon}' \right] \right), \\
\left\{ q(z\epsilon), \overline{q}(z^{-1}\overline{\epsilon}) \right\}_2 (\mathcal{P}) &= Z_{\epsilon,\overline{\epsilon}}.
\end{aligned} \tag{97}$$

Remark 3 *The kink central charge $Z_{\epsilon,\overline{\epsilon}}$ and the $R_\pm$ matrices in the Poisson brackets are related. By construction, the soliton solutions satisfy the constraints $\mu_{l/r} = 0$ and at quantum level, the symmetry superalgebra $\mathfrak{s}$ gets q-deformed $\mathfrak{s} \to \mathfrak{s}_q$. Then, it would be interesting to study the precise relation among the saturation of the quantum Bogomolny bound, the soliton masses, the deformation parameter q and the quantum $R_\pm$ matrices.*

7.2 The superalgebra using the symplectic form Ω_{SSSSG}.

Here we compute the Poisson superalgebra in a different way by using moments and Hamiltonian vector fields as presented in section 6. In particular, we want to verify the result for the mixed bracket (96).

Start with the mixed bracket and consider a contraction of equation (87) with $X_{\overline{\epsilon}} = \left(\delta_{\overline{\epsilon}}\gamma, \delta_{\overline{\epsilon}}\psi, \delta_{\overline{\epsilon}}\overline{\psi} \right)$

$$\Omega_{WZNW}(X_\epsilon, X_{\overline{\epsilon}}) = \Omega_+(X_\epsilon, X_{\overline{\epsilon}}) - \Omega_-(X_\epsilon, X_{\overline{\epsilon}}),$$

where we have to use (82)

$$\begin{aligned}
\delta_{\overline{\epsilon}}\gamma &= \widehat{\overline{\epsilon}}\gamma, \qquad \widehat{\overline{\epsilon}} = \left[\widetilde{\Lambda}\overline{\psi}, \overline{\epsilon} \right] - \overline{w}^\perp, \\
\delta_{\overline{\epsilon}}\overline{\psi} &= \left[\overline{q}^\parallel, \overline{\epsilon} \right] - \left[\overline{w}^\perp, \overline{\psi} \right], \qquad \delta_{\overline{\epsilon}}\psi = - \left[\Lambda, \gamma^{-1}\overline{\epsilon}\gamma \right]
\end{aligned}$$

with $\delta\gamma(X_{\overline{\epsilon}}) = \widehat{\overline{\epsilon}}\gamma$, $\delta\psi(X_{\overline{\epsilon}}) = - \left[\Lambda, \gamma^{-1}\overline{\epsilon}\gamma \right]$, $\delta\overline{\psi}(X_{\overline{\epsilon}}) = \left[\overline{q}^\parallel, \overline{\epsilon} \right] - \left[\overline{w}^\perp, \overline{\psi} \right]$. From (88) we have

$$\begin{aligned}
\Omega_+(X_\epsilon, X_{\overline{\epsilon}}) &= - \left(\widehat{\epsilon}, \delta q(X_{\overline{\epsilon}}) \right) + \left(\epsilon, \left[q, \widetilde{\Lambda}\delta\psi(X_{\overline{\epsilon}}) \right] \right) + \left(w^\perp, \left[\psi, \widetilde{\Lambda}\delta\psi(X_{\overline{\epsilon}}) \right] \right) \\
&= - \left(q^\parallel, \left[\epsilon, \gamma^{-1}\overline{\epsilon}\gamma \right] \right) - \left(\left[\left[\epsilon, \widetilde{\Lambda}\psi \right], \psi \right], \gamma^{-1}\overline{\epsilon}\gamma \right) \\
&= - \left(q, \left[\epsilon, \gamma^{-1}\overline{\epsilon}\gamma \right] \right) + \left(\mu_L, \left[\epsilon, \gamma^{-1}\overline{\epsilon}\gamma \right] \right),
\end{aligned}$$

where we have used $\delta q(X_{\overline{\epsilon}}) = D_+ \left(\gamma^{-1}\delta\gamma(X_{\overline{\epsilon}}) \right) = D_+ \left(\gamma^{-1}\widehat{\overline{\epsilon}}\gamma \right) = \left[\psi, \gamma^{-1}\overline{\epsilon}\gamma \right]$. In a similar manner, we have

$$\begin{aligned}
\Omega_-(X_\epsilon, X_{\overline{\epsilon}}) &= \left(\epsilon, \gamma^{-1}\delta\overline{\psi}(X_{\overline{\epsilon}})\gamma + \left[\gamma^{-1}\overline{\psi}\gamma, \gamma^{-1}\delta\gamma(X_{\overline{\epsilon}}) \right] \right) \\
&= - \left(\overline{q}^\parallel, \left[\gamma\epsilon\gamma^{-1}, \overline{\epsilon} \right] \right) + \left(\gamma\epsilon\gamma^{-1}, \left[\overline{\psi}, \left[\widetilde{\Lambda}\overline{\psi}, \overline{\epsilon} \right] \right] \right) \\
&= - \left(\overline{q}, \left[\gamma\epsilon\gamma^{-1}, \overline{\epsilon} \right] \right) + \left(\mu_R, \left[\gamma\epsilon\gamma^{-1}, \overline{\epsilon} \right] \right).
\end{aligned}$$

Putting all together and restricting to the symplectic quotient (92), we have

$$\Omega_{SSSSG}(X_\epsilon, X_{\overline{\epsilon}}) = - \int_{-\infty}^{+\infty} dx \, \partial_x Str \left(\gamma\epsilon\gamma^{-1}, \overline{\epsilon} \right) = -Z_{\epsilon,\overline{\epsilon}}.$$

After a tedious but straightforward calculation we can show that

$$\Omega_+(X_\epsilon, X_{\epsilon'}) = 2\epsilon \cdot \epsilon' \int_{-\infty}^{+\infty} dx^+ T_{++} + \left(\mu_R, 2\epsilon \cdot \epsilon' q^\perp + \left[w_\epsilon^\perp, w_{\epsilon'}^\perp\right]\right),$$

$$-\Omega_-(X_\epsilon, X_{\epsilon'}) = -2\epsilon \cdot \epsilon' \int_{-\infty}^{+\infty} dx^- T_{-+}$$

and by restriction we get

$$\Omega_{SSSSG}(X_\epsilon, X_{\epsilon'}) = 2\epsilon \cdot \epsilon' \int_{-\infty}^{+\infty} dx \left(T_{++} + T_{-+}\right) = q\left(z^2\left[\epsilon, \epsilon'\right]\right).$$

In a similar way we have

$$\Omega_{SSSSG}(X_{\overline{\epsilon}}, X_{\overline{\epsilon}'}) = 2\overline{\epsilon} \cdot \overline{\epsilon}' \int_{-\infty}^{+\infty} dx \left(T_{+-} + T_{--}\right) = q\left(z^{-2}\left[\overline{\epsilon}, \overline{\epsilon}'\right]\right).$$

Note that the last expressions are independent of the non-local terms present in the field variations (81),(82). This is precisely what we found above in the computation done by using differentials, i.e, the invariance of the brackets (94) and (95) under the action of (25).

Comparing with (97) we have that[29]

$$\{q(a), q(a')\}_2 (\mathcal{P}) = -\Omega_{SSSSG}(X_a, X_{a'}), \qquad a \in \widehat{\mathfrak{s}}.$$

8 Concluding remarks.

We have shown explicitly through simple arguments the existence 2d supersymmetry on the reduced phase space of the GSsσ models on the target spaces $AdS_n \times S^n$, $n = 2, 3, 5$. However, several question are still open and there are some interesting and important directions to be followed in the future. For example, the implementation of the Pohlmeyer reduction on the path integral, e.g. [4], and the study of the supersymmetry in which perhaps, localization techniques can be used. The role of the q-deformed supersymmetry. The supersymmetry associated to the reduction of the GSsσ model on $AdS_4 \times CP^3$. A refine study of the relations between boundary terms and integrability, etc. We hope to address some of these questions in the near future.

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[29]Roughly, if we use (86) and (93) we can write $\{H_a, H_{a'}\}_2 = H_{[a,a']} + c$, i.e, the action $\mathfrak{s} \circlearrowleft \mathcal{P}$ is not Poissonian.

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